Application of Immersed Finite Elements (IFE) to 1D Parabolic Equations with Moving Interface

Slimane Adjerid, Tao Lin, Xu Zhang

Department of Mathematics
Virginia Tech

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FE Methods for Interface Problem

The interface problem

Mathematical model of a physical phenomenon in a domain that consists of multiple materials.

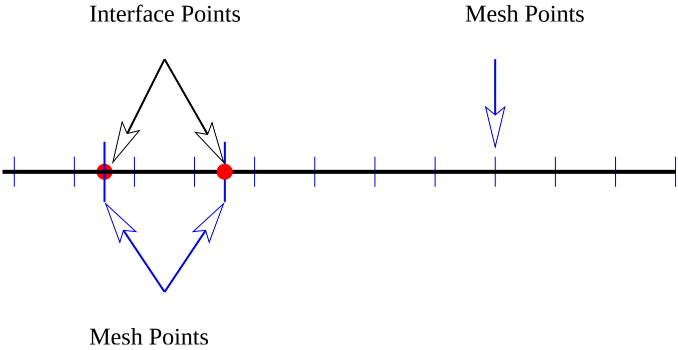
Traditional Finite Element Methods

- Body-fit mesh: meshes are aligned with material interface
- basis function: independent of the interface jump condition

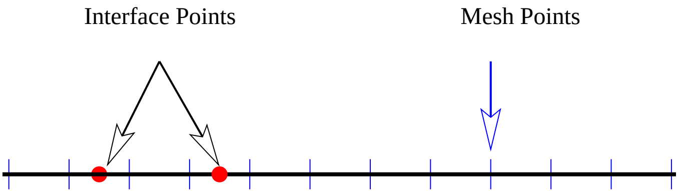
Immersed Finite Element Methods

- Non-body-fit mesh: meshes are independent with material interface (Cartesian Mesh)
- basis function: made to satisfy the interface jump condition

Body-fit meshes for interface problems



Non-body-fit mesh can be use by IFE method



1-D IFE Method Work

- Z. Li, *The immersed interface method using a finite element formulation*, Appl. Numer. Math. 27 (1998) 253 - 267.
- B. Camp, T. Lin, Y. Lin, W.W. Sun, Quadratic immersed finite element spaces and their approximation capabilities, Adv. Comput. Math. 24 (2006) 81 - 112.
- S. Adjerid, T. Lin, *Higher-order immersed discontinuous Galerkin methods*, Internat. J. Inform. Systems Sci. 3 (4) (2007) 555 - 568.
- S. Adjerid, T. Lin, *A p-th degree immersed finite element for boundary value problems with discontinuous coefficients*, Appl. Numer. Math. 59 (2009) 1303 - 1321.

Higher Dimension

This IFE idea has been extended to higher dimensional case.

The parabolic equation with moving interface

The parabolic equation:

$$\begin{array}{ll} u_t(t,x) - (\beta u')' = f, & a < x < b,\ 0 \le t \le T, \\ u(t,a) = u_a(t),\ u(t,b) = u_b(t), & 0 \le t \le T, \\ u(0,x) = u_0(x), & a \le x \le b, \end{array} \tag{1}$$

The discontinuous thermal diffusivity coefficient: $a < \alpha(t) < b$

$$\beta(x) = \begin{cases} \beta^-, & x \in (a, \alpha(t)), \\ \beta^+, & x \in (\alpha(t), b), \end{cases} \tag{2}$$

The jump conditions across the interface : $x = \alpha(t)$

$$[u] = 0, \qquad [\beta u'] = 0 \tag{3}$$

Non-body-fit Mesh Illustration

The extended interface conditions

Smoothness assessment:

$$u_t - (\beta u')' = f, \quad a < x < b, \ 0 \le t \le T$$

$$f \in C\left([0, T]; C^{p-1}(a, b)\right), \qquad p \ge 0$$

$$\Downarrow$$

$$u \in C^1\left([0, T]; C^0(a, b)\right), \qquad \beta u' \in C\left([0, T]; C^p(a, b)\right)$$

The extended interface conditions

Smoothness assessment:

$$u_t - (\beta u')' = f, \quad a < x < b, \ 0 \le t \le T$$

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$$\Downarrow$$

$$u \in C^1\left([0, T]; C^0(a, b)\right), \qquad \beta u' \in C\left([0, T]; C^p(a, b)\right)$$

Extended interface conditions (for $p \ge 2$): $x = \alpha(t)$

$$[u] = 0, \ [\beta u^{(j)}] = 0, \ j = 1, 2, \cdots, p \tag{4}$$

1-D IFE functions

The reference element IFE space

The reference element: $\hat{e} = [-1, 1], \hat{e}^- = [-1, \hat{\alpha}], \hat{e}^+ = [\hat{\alpha}, 1]$

$$\hat{\Pi}_p(\hat{e}) = \{\hat{v} \ : \ \hat{v}|_{\hat{e}^\pm} \in \Pi_p(\hat{e}^\pm), [\hat{v}]_{\hat{\alpha}} = [\beta \hat{v}^{(j)}]_{\hat{\alpha}} = 0, \ j = 1, 2, \cdots, p\},$$

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The reference element: $\hat{e} = [-1, 1], \hat{e}^- = [-1, \hat{\alpha}], \hat{e}^+ = [\hat{\alpha}, 1]$

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$p + 1$ equally space nodes $\hat{e}$:

$$-1 = t_1 < t_2 < t_3 < \cdots < t_p < t_{p+1} = 1.$$

Let $i_{\hat{\alpha}}$ be the integer such that

$$t_{i_{\hat{\alpha}}} < \hat{\alpha} < t_{i_{\hat{\alpha}}+1}.$$

The IFE polynomials:

$\hat{\phi}_i$ for $i = 1, 2, \cdots, i_{\hat{\alpha}}$:

$$\hat{\phi}_i(\hat{x}) = \begin{cases} \hat{\phi}_i^-(\hat{x}) = \hat{\phi}_i^+(\hat{\alpha}) + \frac{\beta^+}{\beta^-}(\hat{\phi}_i^+(\hat{x}) - \hat{\phi}_i^+(\hat{\alpha})), & \hat{x} \in [-1, \hat{\alpha}], \\ \hat{\phi}_i^+(\hat{x}) = \sum_{j=1}^{i_{\hat{\alpha}}} c_j L_j(\hat{x}), & \hat{x} \in [\hat{\alpha}, 1], \end{cases}$$

The IFE polynomials:

$\hat{\phi}_i$ for $i = 1, 2, \cdots, i_{\hat{\alpha}}$:

$$\hat{\phi}_i(\hat{x}) = \begin{cases} \hat{\phi}_i^-(\hat{x}) = \hat{\phi}_i^+(\hat{\alpha}) + \frac{\beta^+}{\beta^-}(\hat{\phi}_i^+(\hat{x}) - \hat{\phi}_i^+(\hat{\alpha})), & \hat{x} \in [-1, \hat{\alpha}], \\ \hat{\phi}_i^+(\hat{x}) = \sum_{j=1}^{i_{\hat{\alpha}}} c_j L_j(\hat{x}), \quad \hat{x} \in [\hat{\alpha}, 1], \end{cases}$$

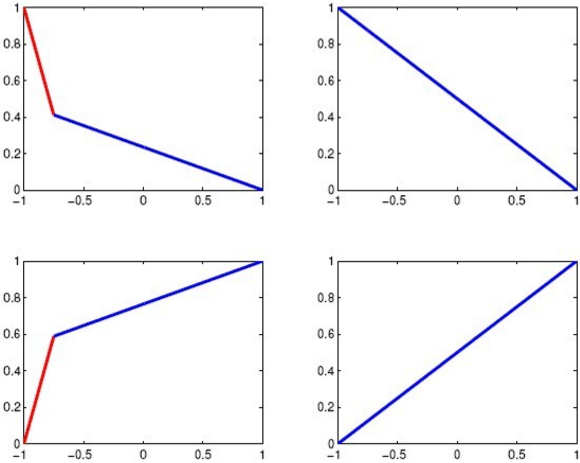
$\hat{\phi}_i$ for $i = i_{\hat{\alpha}} + 1, i_{\hat{\alpha}} + 2, \cdots, p + 1$:

$$\hat{\phi}_i(\hat{x}) = \begin{cases} \hat{\phi}_i^-(\hat{x}) = \sum_{j=1+i_{\hat{\alpha}}}^{p+1} c_j L_j(\hat{x}), \ \hat{x} \in [-1, \hat{\alpha}] \\ \hat{\phi}_i^+(\hat{x}) = \hat{\phi}_i^-(\hat{\alpha}) + \frac{\beta^-}{\beta^+}(\hat{\phi}_i^-(\hat{x}) - \hat{\phi}_i^-(\hat{\alpha})), \ \hat{x} \in [\hat{\alpha}, 1]. \end{cases}$$

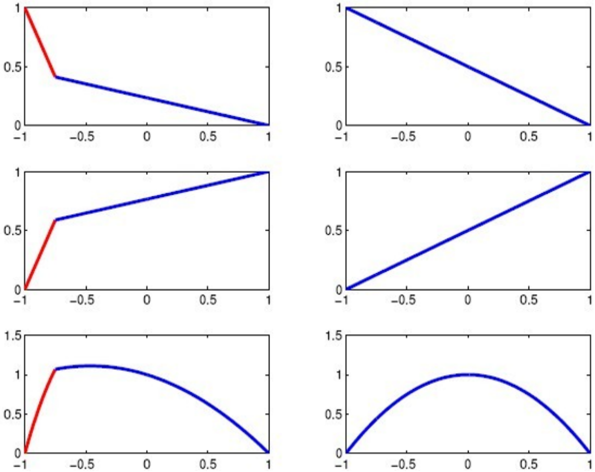
They satisfy the extended jump conditions. The coefficients c_j, $j = 1, 2, \cdots, p+1$ are to be determined by the nodal value constraints:

$$\hat{\phi}_i(\hat{x}_j) = \delta_{ij}, \quad i, j = 1, 2, \cdots, p + 1.$$

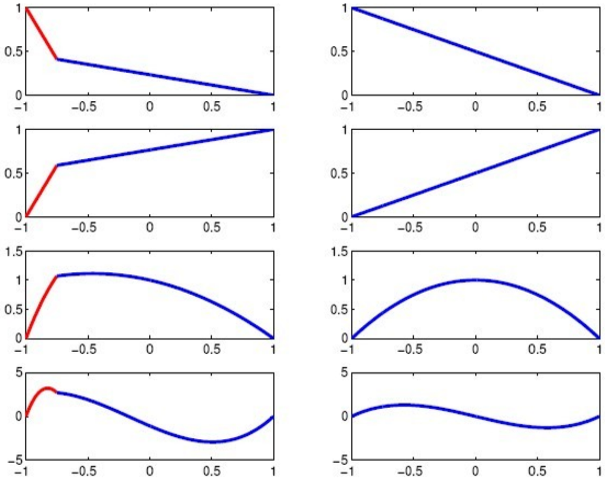
The Linear IFE and Standard basis functions



The Quadratic IFE and Standard basis functions



The Cubic IFE and Standard basis functions



Weak formulation

Let $\Omega = (a, b) \subset \mathbb{R}$. Find $u \in C^1([0, T]; H^1(\Omega))$ such that,

$$\int_\Omega \frac{\partial u}{\partial t} v \, dx - \int_\Omega (\beta u_x)_x v \, dx = \int_\Omega f v \, dx. \quad \forall v \in H_0^1(\Omega)$$

Integration by parts,

$$\int_\Omega \frac{\partial u}{\partial t} v \, dx + \int_\Omega \beta u_x v_x \, dx = \int_\Omega f v \, dx.$$

Simply,

$$(u_t, v) + (\beta u_x, u_x) = (f, v).$$

Time Discretization: Crank-Nicolson

Let k be the time step and u^n the approximation of $u(t,x)$ at $t = t_n = nk$.
Find $u^n \in H^1(\Omega)$, such that

$$\begin{aligned} & \left(\frac{u^n - u^{n-1}}{k}, v^{n-\frac{1}{2}}\right) + \left(\frac{\beta^n u_x^n + \beta^{n-1} u_x^{n-1}}{2}, v_x^{n-\frac{1}{2}}\right) \\ = & \left(f(t_{n-\frac{1}{2}}), v^{n-\frac{1}{2}}\right) \quad \forall v^{n-\frac{1}{2}} \in H_0^1(\Omega). \end{aligned}$$

It is equivalent to

$$\begin{aligned} & (u^n, v^{n-\frac{1}{2}}) + \frac{k}{2}(\beta^n u_x^n, v_x^{n-\frac{1}{2}}) \\ = & (u^{n-1}, v^{n-\frac{1}{2}}) - \frac{k}{2}(\beta^{n-1} u_x^{n-1}, v_x^{n-\frac{1}{2}}) + k(f(t_{n-\frac{1}{2}}), v^{n-\frac{1}{2}}) \end{aligned}$$

Space Discretization: IFE Method

Let $\mathcal{T}_h$ be a structured partition of Ω and let

$$S_h^n = \{\phi_i^n : i = 0, 1, \cdots, N_h, N_h + 1\}$$

be the IFE space at the time $t = t_n = nk$. Here ϕ_0^n, $\phi_{N_h+1}^n$ denote the IFE basis functions on the boundary. So

$$S_{h,0}^n = \{\phi_i^n : i = 1, 2, \cdots, N_h\}$$

denote the IFE space on the interior nodes at the time $t = t_n = nk$

$$u_h^n = \sum_{i=0}^{N_h+1} u_i^n \phi_i^n(x)$$

Space Discretization: IFE Method

Fully discretization Problem

Find $u_h^n \in S_{h,0}^n$, such that,

$$
\begin{aligned}
& (u_h^n, v_h^{n-\frac{1}{2}}) + \frac{k}{2}(\beta^n u_{hx}^n, v_{hx}^{n-\frac{1}{2}}) \\
= \; & (u_h^{n-1}, v_h^{n-\frac{1}{2}}) - \frac{k}{2}(\beta^{n-1} u_{hx}^{n-1}, v_{hx}^{n-\frac{1}{2}}) + k(f(t_{n-\frac{1}{2}}), v_h^{n-\frac{1}{2}}) + b.c.
\end{aligned}
$$

$$\forall v_h^{n-\frac{1}{2}} \in S_{h,0}^{n-\frac{1}{2}}$$

Matrix form:

$$\left(A^{n,n-\frac{1}{2}} + \frac{k}{2}B^{n,n-\frac{1}{2}}\right) U^n = \left(A^{n-1,n-\frac{1}{2}} - \frac{k}{2}B^{n-1,n-\frac{1}{2}}\right) U^{n-1} + kF^{n-\frac{1}{2}}.$$

Error Estimate

Theorem

Let u_h^n and u be solutions of the C-N Galerkin Problem and the original parabolic equation, respectively. Then we have for $n \geq 0$

$$\|u_h^n - u(t_n)\|_{L^2} \leq C(h^{p+1} + k^2).$$

$$\|u_h^n - u(t_n)\|_{H^1} \leq C(h^p + k^2).$$

Error Estimate

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$$\|u_h^n - u(t_n)\|_{H^1} \leq C(h^p + k^2).$$

Proof

Vider Thomée, *Gakerkin Finite Element Methods for Parabolic Problems*

Model Problem

Let $\Omega = (0,1),\ t_0 = 0,\ T = 1,\ \beta^- = 1,\ \beta^+ = 10.$

$$\begin{cases} u^+(t,x) = \left((x-\alpha(t))^{p+1} + \frac{1}{\beta^+}\right)e^x + \left(\frac{1}{\beta^-} - \frac{1}{\beta^+}\right)e^{\alpha(t)} \\ u^-(t,x) = \left((x-\alpha(t))^{p+1} + \frac{1}{\beta^-}\right)e^x \end{cases}$$

where $\alpha(t)$ is the interface moving function. Here we consider two cases:

- $\alpha(t) = \frac{1}{3}t + \frac{1}{3}, \quad 0 \le t \le 1;$
- $\alpha(t) = \frac{1}{4}\sin(2\pi t) + \frac{\pi}{6}, \quad 0 \le t \le 1.$

Interface Moving Illustration: $\alpha(t) = \frac{1}{3}t + \frac{1}{3}$

Linear IFE Method for Example 1: $\alpha(t) = \frac{1}{3}t + \frac{1}{3}$

Error of linear IFE $\|u_h^n - u\|$

h	k	$\|\cdot\|_{L_2}$	$\|\cdot\|_{H_1}$
1/80	1/80	7.4911e-005	1.7281e-002
1/120	1/120	3.1173e-005	1.1302e-002
1/160	1/160	1.8006e-005	8.4670e-003
1/200	1/200	1.1694e-005	6.8109e-003
1/240	1/240	7.9004e-006	5.6348e-003
1/280	1/280	5.8943e-006	4.8279e-003
1/320	1/320	4.5485e-006	4.2396e-003

$$\|u(t_n,\cdot) - u_h^n\|_{L^2} \approx 0.4909h^{2.0113}.$$

$$\|u(t_n,\cdot) - u_h^n\|_{H^1} \approx 1.4458h^{1.0118}.$$

where $t_n = 1$.

Quadratic IFE Method for Example 1: $\alpha(t) = \frac{1}{3}t + \frac{1}{3}$

Error of quadratic IFE $\|u_h^n - u\|$

h	k	$\|\cdot\|_{L_2}$	$\|\cdot\|_{H_1}$
1/16	1/64	3.0238e-005	3.4011e-003
1/25	1/125	6.3537e-006	9.3410e-004
1/36	1/216	2.1227e-006	4.9225e-004
1/49	1/343	7.4729e-007	2.2768e-004
1/64	1/512	3.3450e-007	1.3982e-004
1/81	1/729	1.6114e-007	8.1779e-005
1/100	1/1000	8.4886e-008	5.4826e-005

$$\|u(t_n,\cdot) - u_h^n\|_{L^2} \approx 0.1966h^{3.1920}.$$

$$\|u(t_n,\cdot) - u_h^n\|_{H^1} \approx 1.3518h^{2.2111}.$$

where $t_n = 1$.

Cubic IFE Method for Example 1: $\alpha(t) = \frac{1}{3}t + \frac{1}{3}$

Error of cubic IFE $\|u_h^n - u\|$

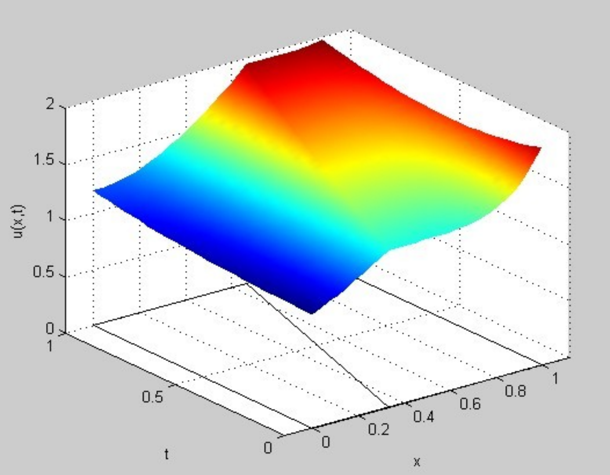
h	k	$\|\cdot\|_{L_2}$	$\|\cdot\|_{H_1}$
1/15	1/225	8.5555e-007	6.0992e-005
1/20	1/400	2.6695e-007	2.7559e-005
1/25	1/625	1.0887e-007	1.3013e-005
1/30	1/900	5.2361e-008	7.4963e-006
1/35	1/1225	2.8490e-008	4.8617e-006
1/40	1/1600	1.6585e-008	3.1671e-006
1/45	1/2025	1.0346e-008	2.2219e-006

$$\|u(t_n,\cdot) - u_h^n\|_{L^2} \approx 0.0448h^{4.0149}.$$

$$\|u(t_n,\cdot) - u_h^n\|_{H^1} \approx 0.2331h^{3.0364}.$$

where $t_n = 1$.

IFE Solution Surface Plot



Interface Moving Illustration: $\alpha(t) = \frac{1}{4}\sin(2\pi t) + \frac{\pi}{6}$

Linear IFE Method for Example 2: $\alpha(t) = \frac{1}{4}\sin(2\pi t) + \frac{\pi}{6}$

Error of linear IFE $\|u_h^n - u\|$

h	k	$\|\cdot\|_{L_2}$	$\|\cdot\|_{H_1}$
1/80	1/80	1.1574e-004	2.1392e-002
1/120	1/120	4.7688e-005	1.4048e-002
1/160	1/160	2.4970e-005	1.0434e-002
1/200	1/200	1.6181e-005	8.3402e-003
1/240	1/240	1.2817e-005	6.9570e-003
1/280	1/280	1.0520e-005	5.9651e-003
1/320	1/320	7.5918e-006	5.2167e-003

$$\|u(t_n,\cdot) - u_h^n\|_{L^2} \approx 0.4847h^{1.9241}.$$

$$\|u(t_n,\cdot) - u_h^n\|_{H^1} \approx 1.8299h^{1.0167}.$$

where $t_n = 1$.

Quadratic IFE Method for Example 2: $\alpha(t) = \frac{1}{4}\sin(2\pi t) + \frac{\pi}{6}$

Error of quadratic IFE $\|u_h^n - u\|$

h	k	$\|\cdot\|_{L_2}$	$\|\cdot\|_{H_1}$
1/16	1/64	2.8665e-004	3.6552e-003
1/25	1/125	7.4826e-005	2.1762e-003
1/36	1/216	2.4547e-005	6.6110e-004
1/49	1/343	9.7443e-006	4.2455e-004
1/64	1/512	4.3892e-006	2.2711e-004
1/81	1/729	2.1475e-006	1.8224e-004
1/100	1/1000	1.1426e-006	8.5876e-005

$$\|u(t_n,\cdot) - u_h^n\|_{L^2} \approx 1.2226h^{3.0155}.$$

$$\|u(t_n,\cdot) - u_h^n\|_{H^1} \approx 1.2000h^{2.0457}.$$

where $t_n = 1$.

Cubic IFE Method for Example 2: $\alpha(t) = \frac{1}{4}\sin(2\pi t) + \frac{\pi}{6}$

Error of cubic IFE $\|u_h^n - u\|$

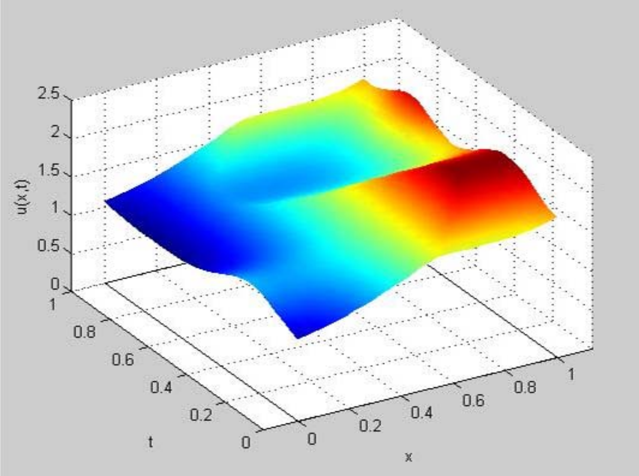
h	k	$\|\cdot\|_{L_2}$	$\|\cdot\|_{H_1}$
1/15	1/225	2.9432e-005	1.4829e-004
1/20	1/400	9.3058e-006	7.1214e-005
1/25	1/625	3.8116e-006	2.4627e-005
1/30	1/900	1.8385e-006	1.5025e-005
1/35	1/1225	9.9228e-007	9.2917e-006
1/40	1/1600	5.8164e-007	5.0762e-006
1/45	1/2025	3.6314e-007	4.2689e-006

$$\|u(t_n,\cdot) - u_h^n\|_{L^2} \approx 1.4911h^{4.0004}.$$

$$\|u(t_n,\cdot) - u_h^n\|_{H^1} \approx 1.4539h^{3.3733}.$$

where $t_n = 1$.

IFE Solution Surface Plot



Conclusion

IFE Method

- The partition is independent of interface. Allow the interface to go through the interiors of elements. (We can use the Cartesian Mesh for interface problem)
- For the moving interface problem, mesh do not need to be regenerated at each time level.
- On the non-interface element, we use standard finite element functions, on the interface element, we use piecewise polynomials
- Similar approximation capability with standard finite element