Improved accuracy with an enhanced physics based scheme for the 3d Navier-Stokes equations

Nick Wilson
newilso@clemson.edu
Department of Mathematical Sciences
Clemson University
Clemson, SC 29634

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Outline

- Introduction to NSE
- Recent work
- Sources of error in computing NSE
- Numerical schemes
- Theorems
- Numerical results

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Navier-Stokes Equations

We consider the 3d Navier-Stokes equation in the domain Ω with Lipschitz continuous boundary.

$$u_t + u \cdot \nabla u + \nabla p - Re^{-1} \Delta u = f, \quad (1)$$

$$\nabla \cdot u = 0. \quad (2)$$

- p is pressure
- u is velocity
- Re is the the Reynolds number
- f if the body force

Energy and helicity

- Recall energy $= \frac{1}{2}\|u\|^2$
  - Energy is a conserved quantity.
  - The nonlinearity preserves it and cascades energy from large to small scales.
  - Known to be of utmost physical importance.
- Helicity $= \int_\Omega u \cdot (\nabla \times u) dx$
  - Physical interpretation: The degree to which vortex lines are tangled and intertwined (Moffatt, Tsoniber 1992).
  - Topological interpretation: H = 0 iff velocity field is reflectionally symmetric (MT 1992).
  - Helicity is also a conserved quantity.
  - The nonlinearity also cascades helicity, H(k)$\approx k^{-\frac{5}{3}}$ (Ditlevson, Guiliani 2001) (Q.Chen, S.Chen,G.Eyink 2003).

Motivation for a scheme to conserve helicity

- Helicity is important physically; it seems intuitive that a numerical scheme will perform better if it correctly accounts for helicity (as well as energy).
- This idea is not unprecedented. *Enhanced physics based schemes* have provided more accurate simulations, especially over longer time intervals.
- Arakawa's energy and enstrophy conserving scheme for the 2D NSE.
- For shallow water equations, Arakawa/Lamb's energy and potential enstrophy conserving scheme (1981).
- For 3d axisymmetric flow J.G Liu and Wang introduced an energy and helicity conserving scheme (2004).

A scheme that conserves energy and helicity

- For full 3d NSE Rebholz introduced a finite element scheme that conserved energy and helicity with periodic boundary conditions (2007).

$$\begin{aligned} (\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h) - (p_h^{n+\frac{1}{2}}, \nabla \cdot v_h) & & \\ +(w_h^{n+\frac{1}{2}} \times u_h^{n+\frac{1}{2}}, v_h) + Re^{-1}(\nabla u_h^{n+\frac{1}{2}}, \nabla v_h) &=& (f(t^{n+\frac{1}{2}}), v_h) \\ (\nabla \cdot u_h^{n+1}, q_h) &=& 0 \\ (w_h^{n+1}, \chi_h) + (\lambda_h^{n+1}, \nabla \cdot \chi_h) &=& (\nabla \times u_h^{n+1}, \chi_h) \\ (\nabla \cdot w_h^{n+1}, r_h) &=& 0 \end{aligned}$$

$\forall (v_h; q_h; \chi_h; r_h) \in (X_h; Q_h; X_h; Q_h)$.

Objective and complication

- We want to extend the scheme to more general boundary conditions.
- Possible drawback: uses Bernoulli pressure $P = p + \frac{1}{2}|u|^2$.
    - More complex than usual pressure.
    - Contains boundary layers.
- In a stable finite element scheme velocity is often approximated by polynomials of degree k, where pressure is approximated by polynomials of degree (k-1).
- Numerically this creates large pressure errors, which can adversely affect the velocity error.
- vel error $\approx$ Re*Bernoulli pressure

An idea of how to reduce error

- Layton/Manica/Neda/Olshanskii/Rebholz, have shown how grad-div stabilization can improve velocity error by reducing the effect of pressure error.
  - Error is scaled by $Re^{\frac{1}{2}}$ instead of Re.
  - Add $(\nabla \cdot u_h^{n+\frac{1}{2}}, \nabla \cdot v_h)$ to left hand side of scheme.
  - This is consistent since it is derived by adding the identically zero term $-\nabla(\nabla \cdot u)$ to the continuous NSE.
  - Penalizes for lack of mass conservation.
- This effect of the grad div term was first noted by Olshanskii for Stokes equations (2002).
- We consider this term as well as $-\nabla(\nabla \cdot u_t)$.
- Gives better energy balance and reduces the effect of pressure error.
- Not as strong a penalization for lack of mass conservation.

The schemes

- The goal of this work is to combine the extension of the energy helicity conserving scheme to Dirichlet boundary conditions with necessary stabilizations.
- We will consider 3 numerical schemes of the form

$$(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h) - (P_h^{n+\frac{1}{2}}, \nabla \cdot v_h) + STAB$$
$$(w_h^{n+\frac{1}{2}} \times u_h^{n+\frac{1}{2}}, v_h) + (\nabla u_h^{n+\frac{1}{2}}, \nabla v_h) = (f(t^{n+\frac{1}{2}}), v_h).$$

- $STAB_1 = 0$
- $STAB_2 = (\nabla \cdot u_h^{n+\frac{1}{2}}, \nabla \cdot v_h)$
- $STAB_3 = \frac{1}{\Delta t}(\nabla \cdot (u_h^{n+1} - u_h^n), v_h)$

Energy conservation

- Energy conservation without stabilization

$$\frac{1}{2}\|u_h^M\|^2 + \nu\Delta t \sum_{n=0}^{M-1} \|\nabla u_h^{n+\frac{1}{2}}\|^2 = \Delta t \sum_{n=0}^{M-1} (f(t^{n+\frac{1}{2}}), u_h^{n+\frac{1}{2}}) + \frac{1}{2}\|u_h^0\|^2$$

- The use of *$STAB_2$*and *$STAB_3$*adds the following left hand side terms respectively

$$\Delta t \sum_{n=0}^{M-1} ||\nabla \cdot u_h^{n+\frac{1}{2}}||^2,$$

$$\frac{1}{2}(||\nabla \cdot u_h^M||^2 - ||\nabla \cdot u_h^0||^2).$$

Helicity conservation laws

Scheme 1:

$$H_h^M + 2\nu\Delta t \sum_{n=0}^{M-1} (\nabla u_h^{n+\frac{1}{2}}, \nabla w_h^{n+\frac{1}{2}}) = 2\nu\Delta t \sum_{n=0}^{M-1} (f(t^{n+\frac{1}{2}}), \nabla w_h^{n+\frac{1}{2}}) + H_h^0$$

- $STAB_2$ adds to the left hand side the term

$$2\Delta t \sum_{n=0}^{M-1} (\nabla \cdot u_h^{n+\frac{1}{2}}, \nabla \cdot w_h^{n+\frac{1}{2}}).$$

- $STAB_3$ adds to left hand side the term

$$2\Delta t \sum_{n=0}^{M-1} (\nabla \cdot (u_h^{n+1} - u_h^{n}), \nabla \cdot w_h^{n+\frac{1}{2}}).$$

Convergence

Under usual assumptions,

$$\|u(T) - u_h^M\|^2 + Re^{-1}\Delta t \sum_{n=0}^{M-1} \|\nabla(u(t^{n+\frac{1}{2}}) - u_h^{n+\frac{1}{2}})\|^2$$

$$\leq C(\nu^{-1}\Delta t^4 + Re\Delta t \sum_{n=0}^{M} \inf_{q_h \in Q_h} \|P(t^{n+\frac{1}{2}}) - q_h\|^2$$

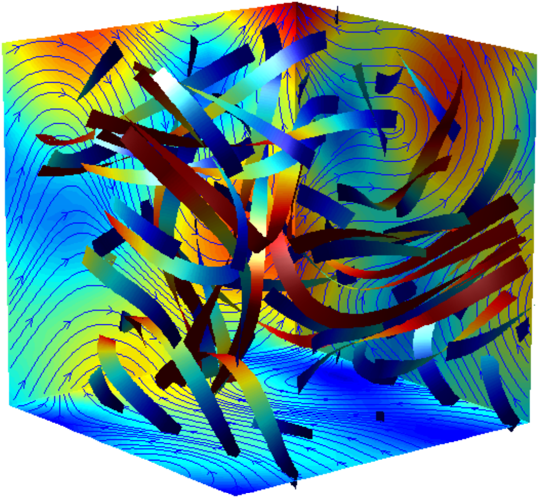
$$+ Re\Delta t \sum_{n=0}^{M} \inf_{v_h \in X_h} \|\nabla(u(t^{n+\frac{1}{2}}) - v_h)\|^2)$$

- The use of stabilization allows for rescaling of $\inf_{q_h \in Q_h} \|P(t^{n+\frac{1}{2}}) - q_h\|^2$ by 1 instead of Re.

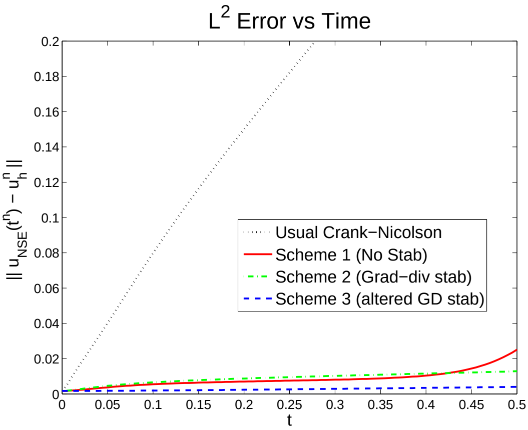
Numerical Test

- We now test the schemes on the Either-Steinman problem on $(-1,1)^3$, with $Re = 1,000$.
- The problem has nontrivial helicity and complex flow structure.
- Additionally, we have computed solutions using the standard Crank-Nicolson scheme.
- We will compare the velocity and helicity errors of the computed solutions.

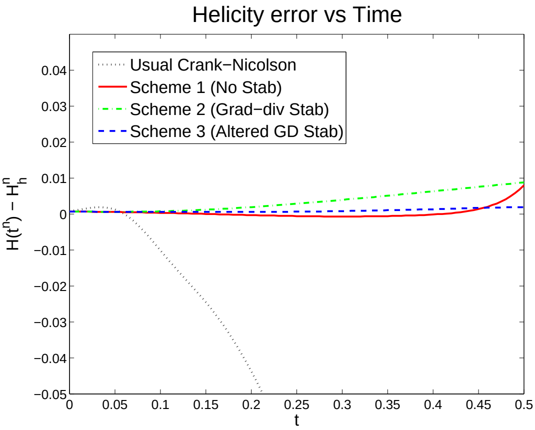
Flow Structure



Velocity Error



Helicity Error



Conclusions and future work

- The use of grad-div stabilization improved accuracy in the energy helicity conserving scheme.
- Altered grad-div stabilization gives slightly better results than the usual grad-div stabilization on our example.
- Additionally the altered grad-div stabilization gives better analytical physical properties.

- Test on more physical problems.
- Using Scott-Vogelius elements.