Two-Level Discretizations of Closure Models for Proper Orthogonal Decomposition

Zhu Wang

Department of Mathematics
Virginia Tech
wangzhu@vt.edu

SIAM Student Conference 2010, VT
Feb 20, 2010

VirginiaTech
Invent the Future

Collaborators

Imran Akhtar
Interdisciplinary Center for Applied Mathematics
Virginia Tech
akhtar@vt.edu

Jeff Borggaard
Department of Mathematics
Interdisciplinary Center for Applied Mathematics
Virginia Tech
jborggaard@vt.edu

Traian Iliescu
Department of Mathematics
Interdisciplinary Center for Applied Mathematics
Virginia Tech
iliescu@vt.edu

POD-Galerkin and Energy Cascade

POD: $\mathbf{u}(x,t) \approx \mathbf{u}^r = \bar{\mathbf{u}}(x) + \sum_{\mathbf{n}=1}^{r} a_{\mathbf{n}}(t)\phi_{\mathbf{n}}(x)$

POD-Galerkin

$$\left(\frac{\partial \mathbf{u}^r}{\partial t}, \phi_{\mathbf{n}}\right) + ((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \phi_{\mathbf{n}}) + \left(\frac{2}{Re}\mathbb{D}(\mathbf{u}^r), \nabla\phi_{\mathbf{n}}\right) = 0, \quad \mathbf{n} = 1, \cdots, r$$

- energy cascade $\Longrightarrow$ eddy viscosity
- *Couplet, Sagaut, Basdevant, J. Fluid Mech., 2003*

VirginiaTech Invent the Future

Mixing Length Model

- *Aubry, Holmes, Lumley, Stone, J. Fluid Mech., 1988*

$$\left(\frac{\partial \mathbf{u}^r}{\partial t}, \phi_{\mathbf{n}}\right) + \left((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \phi_{\mathbf{n}}\right) + \left(\left(\nu_{ML} + \frac{2}{Re}\right) \mathbb{D}(\mathbf{u}^r), \nabla \phi_{\mathbf{n}}\right) = 0, \quad \mathbf{n} = 1, \cdots, r$$

- $\nu_{ML} = \alpha\, \nu_T = \alpha\, U_{ML}\, L_{ML}$
- *Podvin, Lumley, J. Fluid Mech., 1998*

VirginiaTech
Invent the Future

LES-POD

- Smagorinsky type

$$\left(\frac{\partial \mathbf{u}^r}{\partial t}, \phi_{\mathbf{n}}\right) + ((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \phi_{\mathbf{n}}) + \left(\left(\nu_S + \frac{2}{Re}\right) \mathbb{D}(\mathbf{u}^r), \nabla \phi_{\mathbf{n}}\right) = 0, \quad \mathbf{n} = 1, \cdots, r$$

- $\nu_S := C_S \, |\mathbb{D}(\mathbf{u}^r)|$

Virginia Tech Invent the Future

LES-POD

Smagorinsky type LES-POD on N-S equations

$$(\mathbf{u}^r_t, \phi) + ((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \phi) + ((2Re^{-1} + C_s\|\mathbb{D}(\mathbf{u}^r)\|)\mathbb{D}(\mathbf{u}^r), \nabla\phi) = 0$$

ODE System $\dot{a} = A + \widetilde{A} + (B + \widetilde{B})a + a^T C a;$

$$\frac{\mathrm{d}a_k(t)}{\mathrm{d}t} = \mathcal{A}_k + \widetilde{\mathcal{A}}_k + \sum_{m=1}^r (\mathcal{B}_{km} + \widetilde{\mathcal{B}}_{km}) a_m(t) + \sum_{m=1}^r \sum_{n=1}^r \mathcal{C}_{kmn} a_n(t) a_m(t)$$

- $\mathcal{A}_k = -(\phi_k, \bar{\mathbf{u}} \cdot \nabla\bar{\mathbf{u}}) - 2Re^{-1}(\nabla\phi_k, \frac{\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T}{2}),$
- $\mathcal{B}_{km} = -(\phi_k, \bar{\mathbf{u}} \cdot \nabla\phi_m) - (\phi_k, \phi_m \cdot \nabla\bar{\mathbf{u}}) - 2Re^{-1}(\nabla\phi_k, \frac{\nabla\phi_m + \nabla\phi_m{}^T}{2}),$
- $\mathcal{C}_{kmn} = -(\phi_k, \phi_m \cdot \nabla\phi_n).$

- $\widetilde{\mathcal{A}}_k = -2C_s(\nabla\phi_k, \|\mathbb{D}(\bar{\mathbf{u}} + \sum_{n=1}^r a_n(t)\phi_n(x))\| \frac{\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T}{2}),$
- $\widetilde{\mathcal{B}}_{km} = -2C_s(\nabla\phi_k, \|\mathbb{D}(\bar{\mathbf{u}} + \sum_{n=1}^r a_n(t)\phi_n(x))\| \frac{\nabla\phi_m + \nabla\phi_m{}^T}{2}).$

LES-POD

Smagorinsky type LES-POD on N-S equations

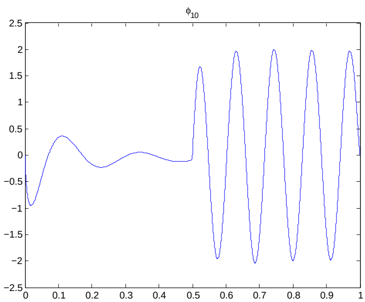
$$(\mathbf{u}^r_t, \phi) + ((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \phi) + ((2Re^{-1} + C_s \|\mathbb{D}(\mathbf{u}^r)\|)\mathbb{D}(\mathbf{u}^r), \nabla\phi) = 0$$

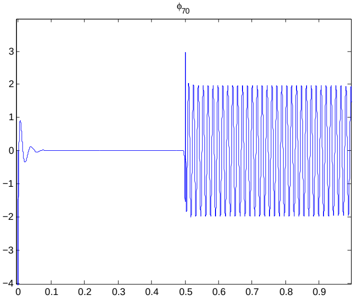
ODE System $\dot{a} = A + \widetilde{A} + (B + \widetilde{B})a + a^T C a;$

$$\frac{\mathrm{d}a_k(t)}{\mathrm{d}t} = \mathcal{A}_k + \widetilde{\mathcal{A}}_k + \sum_{m=1}^{r} (\mathcal{B}_{km} + \widetilde{\mathcal{B}}_{km}) a_m(t) + \sum_{m=1}^{r} \sum_{n=1}^{r} \mathcal{C}_{kmn} a_n(t) a_m(t)$$

- $\mathcal{A}_k = -(\phi_k, \bar{\mathbf{u}} \cdot \nabla\bar{\mathbf{u}}) - 2Re^{-1}(\nabla\phi_k, \frac{\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T}{2}),$
- $\mathcal{B}_{km} = -(\phi_k, \bar{\mathbf{u}} \cdot \nabla\phi_m) - (\phi_k, \phi_m \cdot \nabla\bar{\mathbf{u}}) - 2Re^{-1}(\nabla\phi_k, \frac{\nabla\phi_m + \nabla{\phi_m}^T}{2}),$
- $\mathcal{C}_{kmn} = -(\phi_k, \phi_m \cdot \nabla\phi_n).$
- $\widetilde{\mathcal{A}}_k = -2C_s(\nabla\phi_k, \|\mathbb{D}(\bar{\mathbf{u}} + \sum_{n=1}^{r} a_n(t)\phi_n(x))\| \frac{\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T}{2}),$
- $\widetilde{\mathcal{B}}_{km} = -2C_s(\nabla\phi_k, \|\mathbb{D}(\bar{\mathbf{u}} + \sum_{n=1}^{r} a_n(t)\phi_n(x))\| \frac{\nabla\phi_m + \nabla{\phi_m}^T}{2}).$

VirginiaTech Invent the Future





ϕ_{10} ϕ_{70}

- $\phi_{10} \longrightarrow$ mesh size H
- $\phi_{70} \longrightarrow$ mesh size h
- $h \ll H$

Two-Level POD-EV algorithms

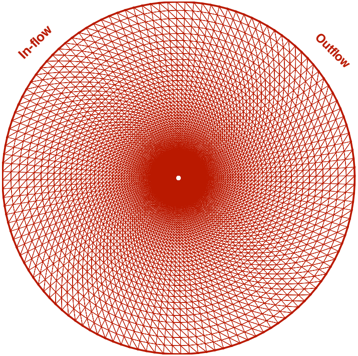
- Temporal discretization - forward Euler method;
- Spatial discretization - finite element method.

(I) Two-level Coarse POD-EV algorithm

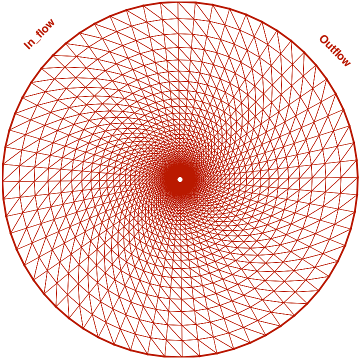
$k = 0$; compute $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on <u>coarse</u> mesh ;
for $k = 1$ to M
  compute $\widetilde{\mathcal{A}}_k, \widetilde{\mathcal{B}}_k$ on <u>coarse</u> mesh ;
  $\mathbf{a}^{k+1} := F(\mathbf{a}^k)$;
endfor

(II) Two-level Hybrid POD-EV algorithm

$k = 0$; compute $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on <u>fine</u> mesh ;
for $k = 1$ to M
  compute $\widetilde{\mathcal{A}}_k, \widetilde{\mathcal{B}}_k$ on <u>coarse</u> mesh ;
  $\mathbf{a}^{k+1} := F(\mathbf{a}^k)$;
endfor

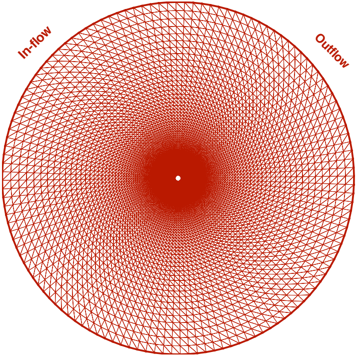


Fine mesh



Coarse mesh, $R_c = 2$

Experiment 1. 2D flow passed the cylinder

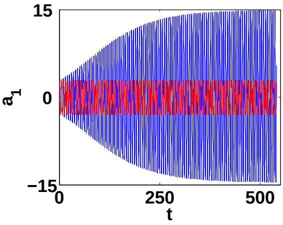


- Re = 200;
- Diameter(cylinder) = D;
- Diameter(Domain) = 50D;
- Grid size (fine) - 192$\times$ 256;
- Inflow B. C. - Dirichlet;
- Outflow B. C. - Neumann;
- Surface of cylinder - No-slip;
- 40 snapshots.

VirginiaTech
Invent the Future

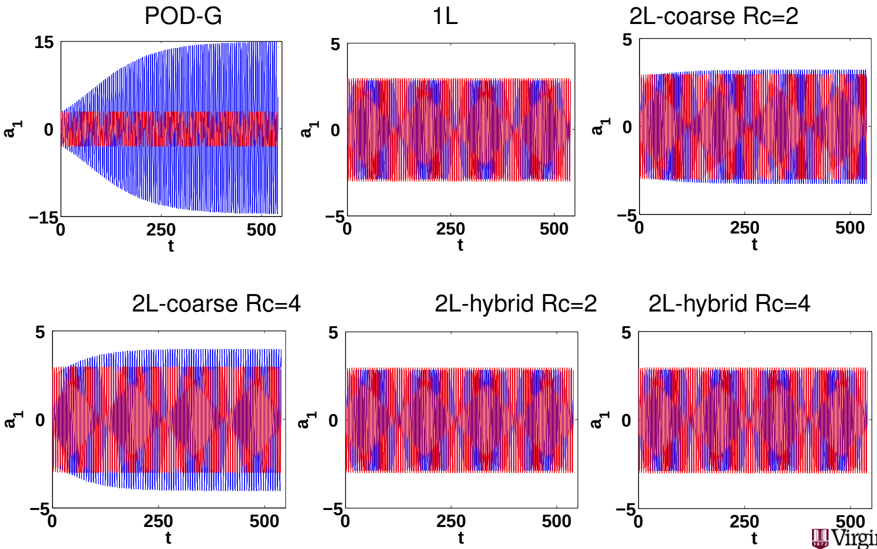
POD-Galerkin

- red - DNS projection
- blue - POD basis coefficients (r=4)



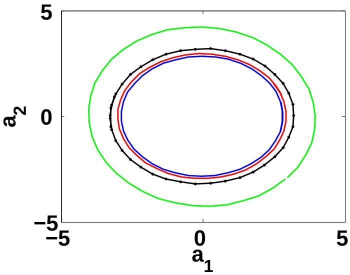
DNS POD-G

- red - DNS projection;
- blue - POD basis coefficients.

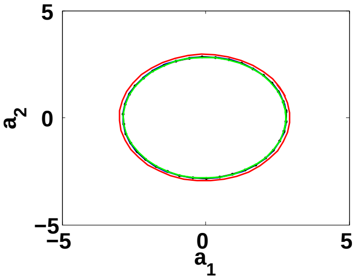


Two-Level POD-EV

- red - DNS projection;
- black -2L POD-EV, $R_c = 2$;
- blue - 1L POD-EV;
- green -2L POD-EV, $R_c = 4$.



2L-coarse



2L-Hybrid

Average error over 100 shedding cycles

Time average error

$$error = \frac{\frac{1}{m}\sum_{k=1}^{m} \|\mathbf{u}^{POD}(\mathbf{x},t_k) - \mathbf{u}^{DNS}(\mathbf{x},t_k)\|_0^2}{\frac{1}{m}\sum_{k=1}^{m} \|\mathbf{u}^{DNS}(\mathbf{x},t_k)\|_0^2}$$

1L POD-EV		R_c	2L coarse POD-EV		2L hybrid POD-EV	
CPU-time (s)	*error*		CPU-time (s)	*error*	CPU-time (s)	*error*
		2	9.94e+3	0.0102	1.01e+4	0.0090
4.01e+4	0.0090	4	2.52e+3	0.0112	2.75e+3	0.0090

- R_c=4: $error_{1L} \approx error_{2L}$, $Time_{1L} \approx 15 Time_{2L}$

VirginiaTech Invent the Future

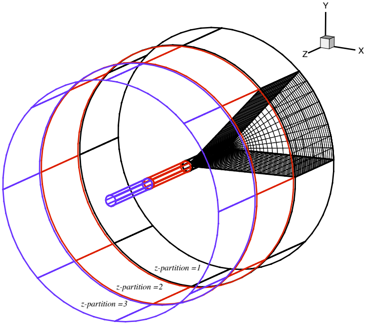
DNS

1L

2L-Coarse Rc=2

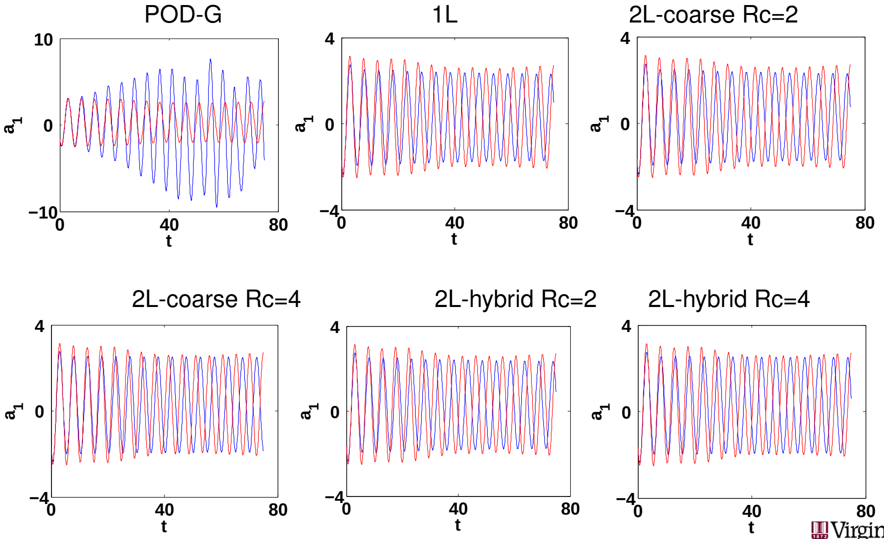
2L-Hybrid Rc=4

Experiment 2. 3D flow passed the cylinder



- Re = 1000;
- Diameter(cylinder) = D;
- Diameter(Domain) = 15D;
- Grid size (fine) - $145 \times 193 \times 17$;
- Inflow B. C. - Dirichlet;
- Outflow B. C. - Neumann;
- Surface of cylinder - No-slip;
- 1000 snapshots.

- red - DNS projection;
- blue - POD basis coefficients (r=6).



VirginiaTech
Invent the Future

Average error

1L POD-EV		R_c	2L coarse POD-EV		2L hybrid POD-EV	
CPU-time (s)	*error*		CPU-time (s)	*error*	CPU-time (s)	*error*
		2	1.07e+5	0.0444	1.02e+5	0.0452
5.32e+5	0.0446	4	2.17e+4	0.0385	2.20e+4	0.0473

- R_c=4: $error_{1L} \approx error_{2L}$, $Time_{1L} \approx 25 Time_{2L}$

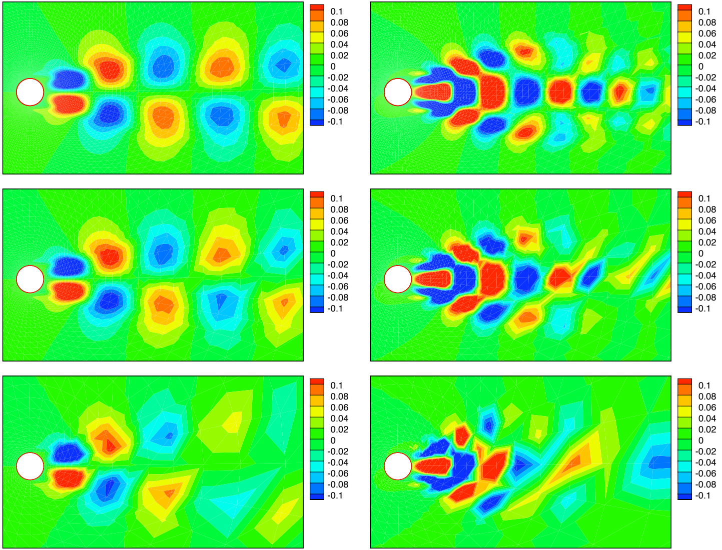
VirginiaTech
Invent the Future

Conclusions

- more physical closure models
- two-level discretization
- $error_{2L} \approx error_{1L}$
- $Time_{1L} = Time_{2L} \times \mathcal{O}(10)$

Thank You!

Virginia Tech
Invent the Future


0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1