Parameter Estimation for Linear Stochastic Elliptic Partial Differential Equations

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Uncertainty Quantification

Physical Systems are affected by uncertainties

- Aleatoric Uncertainty (inherent variability in system parameters/ operating conditions):
  - i) Incomplete Data (porosity in aquifers/ oil reservoirs)
  - ii) Not all scales (in data/solution) are resolved
- Epistemic Uncertainty (Modelling error due to lack of knowledge about the physical system)

Question: What is the impact of model uncertainty on the outputs of the model?

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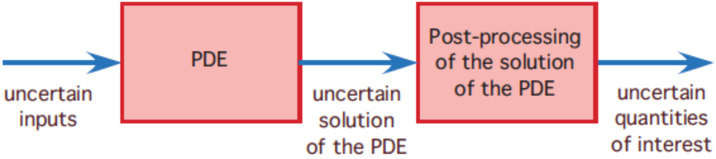
Systems governed by PDEs

Uncertainties in physical processes can occur as:

- random coefficients and forcing terms
- random boundary conditions or initial conditions
- random geometry



Linear Elliptic Stochastic PDEs

A parametrized family of deterministic PDEs:

Let $(\Omega, P, \mathcal{F})$ be a complete probability space and D be a physical domain. Find u satisfying:

$$\begin{aligned} \nabla \cdot (a(x,\omega)\nabla u(x.\omega)) &= f(x,\omega) && \text{on } D \times \Omega \\ u(x,\omega) &= 0 && \text{on } \partial D \times \Omega \end{aligned}$$

Variational Formulation:

Find $u \in \tilde{H}_0^1 := H_0^1(D) \otimes L_P^2(\Omega)$ so that

$$E\left[\int_D a(x,\omega)\nabla u(x,\omega)\nabla v(x,\omega)dx\right] = E\left[\int_D f(x,\omega)v(x,\omega)dx\right]$$

for all $v \in \tilde{H}_0^1$

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Modelling Uncertain Parameters Probalistically

No Spatial Dependance:

- Model input data using random variables

Spatial Dependance (random fields):

- White Noise - data vary randomly and independantly from one point in D to another (Thermal fluctuation/surface roughness)
- Colored Noise - additional correlation structure (bone densities, permeabilities within subsurface layers)

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The Karhunen-Loève Expansion

If the random field $a(x, \omega) \in L^2(D) \otimes L^2_P(\Omega)$ then it is fully characterized by its mean $\bar{a}(x)$ and covariance $cov(x, x')$ via the KL expansion:

$$a(x, \omega) = \bar{a}(x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} b_n(x) Y_n(\omega)$$

where

- $\{b_n, \lambda_n\}$ are eigenpairs of the covariance mapping $(\lambda_1 \geq \lambda_2 \geq ... > 0)$
- $\{Y_n\}$ are centered, uncorrelated random variables

The Finite Noise Assumption

The decay rate of the eigenvalues λ_n depends on

- the smoothness of the covariance kernel and
- is well understood (Schwab Todor [5]).

$\Rightarrow$ Approximate $a(x,\omega)$ using the truncated KL expansion.

$$a(x,\omega) \approx \bar{a}(x) + \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)$$

or

$$a(x,\omega) = a_{min} + e^{\hat{a}(x) + \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)}$$

- Problem now only depends on a finite number of random degrees of freedom.
- Doob-Dynkin Lemma $\Rightarrow$ solution $u(x,\omega) = u(x, Y_1, ..., Y_n)$.

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Stochastic Galerkin Methods

Variational formulation reduces to: find $u(x, y)$ so that for all $v \in \tilde{H}_0^1(D)$

$$\int_\Omega \int_D \nabla u(x, \bar{y}) \nabla v(x, \bar{y}) \rho(\bar{y}) dx d\bar{y} = \int_\Omega \int_D f(x, \bar{y}) v(x, \bar{y}) \rho(\bar{y}) dx d\bar{y}$$

where $\bar{y} = (y_1^a, ..., y_{N_a}^a, y_1^f, ..., y_{N_f}^f)$.

- **Spatial Discretization: standard finite element**
- **Discretization in Probability space...**

$$u^N(x, \omega) = \sum_{j=1}^{J} \sum_{k=1}^{K} c_{jk} \phi_j(x) \psi_k(\bar{y})$$

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Discretisation in Probability Space

- Full Orthogonal Polynomial Spaces- Polynomial Chaos (Ghanem, Spanos [3])
  - Leads to fully coupled $JK \times JK$ system the curse of dimensionality $\Rightarrow$ can't make use of existing solvers.
  - Non-intrusive PCE uncoupling (requires independance)
- Sparse Grid Collocation (Webster, Nobile, Tempone [4])
  - Can make use of legacy codes (parallelisable)
  - Anisotropic collocation methods, adaptive quadrature...
- Other Sampling Methods (Monte Carlo etc.)
  - For highly uncertain systems.

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Experimental Study: Vibration Response of a Beam with Random mass Distribution

Adhikari, Lonkar, Friswell
(University of Bristol) [1]

Figure: Experimental Setup

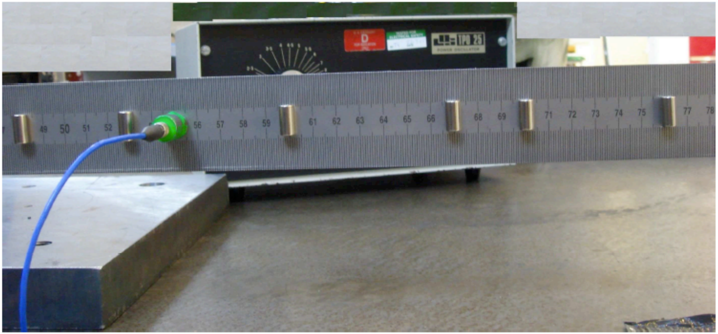
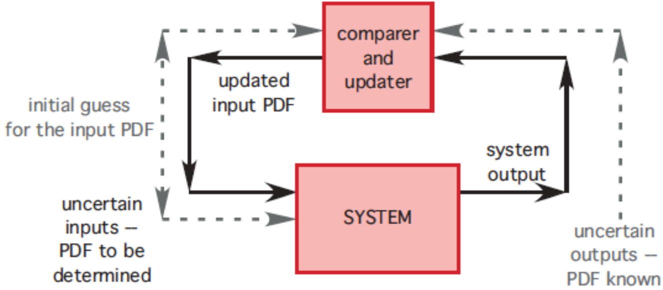


Figure: Randomly positioned weights

The Related Inverse Problem

Given statistical measurements of the outputs of the system, is it possible to derive quantities of interest of the system inputs?



Model calibration – inverse problem

Least Squares Formulation

Consider the Linear Elliptic SPDE

$$\begin{aligned} \nabla \cdot (a(x,\omega)\nabla u(x.\omega)) &= f(x,\omega) && \text{on } D \times \Omega \\ u(x,\omega) &= 0 && \text{on } \partial D \times \Omega \end{aligned}$$

where $f(x,\omega)$ is given. Suppose further that physical measuments $\hat{u}$ are taken either

- at discrete points in D (random variables) or
- in the form of distributed data (random field)

Find the diffusion coefficient $a(x,\omega)$ so as to minimize the cost functional.

$$J(a) := E[\|u(x,.) - \hat{u}(x,.)\|^2 + \beta\|\nabla u(x,.)\|^2]$$

Current Existing Results

Zabaras, Narayanan: *Inverse heat conduction using a spectral approach* [2]:
Measurements of the output are modelled using Generalized Polynomial Chaos Expansion - critiques $\Rightarrow$

- 'Unphysical' measurement.
- Considerable effort (and software development) required to reformulate validated direct-, sensitivity- and adoint equations in the stochastic setting.
- Coupling leads to untenable algorithm when the number of uncertain parameters increases.

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Parameter ID using Stochastic Collocation methods

Zabaras, Ganapathysubramanian: *A scalable framework for the solution of stochastic inverse problems using a sparse grid collocation approach* [6].

- Gradient- based, scalable (w.r.t. sources of uncertainty).
- Non-intrusive (makes use of existing deterministic solvers)
- Promising Numerical Results

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Algorithm

Suppose we have random measurements $\hat{u}(x_1, \omega)$, $\hat{u}(x_2, \omega)$,...,$\hat{u}(x_n, \omega)$).
Approximate $a \approx \sum_{i=1}^{m} a(x, y_i)L(y_i)$, where y_i are fixed collocation points. Now find the m deterministic functions $a(x, y_i)$ minimizing the cost functional:

$$J(a) = \sum_{j=1}^{n} \sum_{k=1}^{p} (E[u(x_j)^k] - E[\hat{u}(x_j)^k])^2$$

(Moment Matching). The sensitivity equations satisfied by the directional derivative of u w.r.t $\{a(., y_i)\}$ are derived formally and used (in a steepest descent algorithm) to update the cost functional .

- Heuristic Approach, No guarantee of existence of solutions.
- No convergence analysis...

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Current Work

Existence of Minimizers for elliptic Problem:

Theorem

If in the above framework f is assumed to be in $L^2 \otimes L^2$ then the problem

$$\textbf{Minimize}_{a \in \tilde{Q}} J(a) := E[\|u(x,\omega) - \hat{u}(x,\omega)\|_Z^2 + \beta \|\nabla u\|^2]$$

where $\tilde{Q} = \{a \in H^2 \otimes L^2 : \|a\|_{\tilde{H}^2} \leq K, 0 < a_{min} \leq a(x.\omega) \leq a_{max} in D \times \Omega\}$ has a unique solution.

PROVED

Future Work

- To prove the approximability of solutions under discretization
- The Existence of Lagrange Multipliers
- Derivation of Optimality system to estimate first p Moments.
- Numerical Implementation

Thanks to

Dr. Jeff Borggaard, Dr. Max Gunzburger

Diagrams due to Dr. M. Gunzburger

Questions?

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