Analysis and approximation of the velocity tracking problem for the Navier-Stokes/Brinkman model

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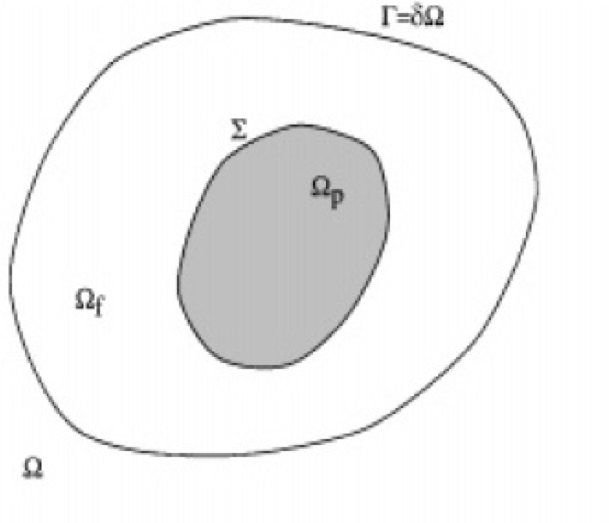
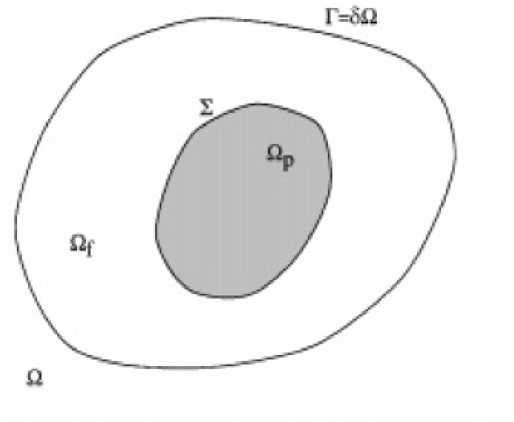
Outline

- Introduction to the model
- The optimal control problem
- Semidiscrete-in-time approximation
- Fully discrete time-space approximation
- Future work

Introduction to the model



- Ω_f: Navier-Stokes equation
- Ω_p: Darcy's law



Navier-Stokes/Brinkman equation

Find (u,v) in Ω such that

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p + \frac{\mu}{K} u = f$$
$$\nabla \cdot u = 0$$

Arquis, Caltagirone (1984)

$L = \frac{\mu}{K}$

$L_g(x) := L(x - g)$

Navier-Stokes/Brinkman Model

Find (u, v) *in* Ω *such that*

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p + L_g u = f$$
$$\nabla \cdot u = 0$$

Problem: Find $g \in \Omega$ such that the solution maximizes some performance

Micrositing

The optimal control problem

$L_g : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$

State equation

$$\begin{cases} u_t + u \cdot \nabla u - \nu \Delta u + \nabla p + L_g u = f \text{ in } \Omega \\ \nabla \cdot u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

$\mathcal{U}_{ad} = \{(u,g)$ satisfies NS/Brinkman, $g \in \Omega\}$

Cost function:

$$J(u,g) = \frac{\alpha}{2} \int_0^T \int_\Omega L_g |u - U|^2 dx dt + \frac{\delta}{2} |g - G|^2$$

Optimal control problem: Find min $J(u,g)$, $(u,g) \in \mathcal{U}_{ad}$

Theorem (Existence) There exist $(u,g) \in \mathcal{U}_{ad}$ that is global minimizer of the cost function.

The optimality system

Navier-Stokes/Brinkman system

$$\begin{cases} u_t + u\cdot\nabla u - \nu\Delta u + \nabla p + L_g u = f \\ \nabla\cdot u = 0 \end{cases}$$

with initial condition: $u(x,0) = u_0(x)$

Adjoint system

$$\begin{cases} -\widehat{w}_t + (\nabla u)^T\cdot\widehat{w} - (u\cdot\nabla)\widehat{w} - \nu\Delta\widehat{w} + \nabla q + L_g^T\widehat{w} = \alpha L_g(u-U) \\ \nabla\cdot\widehat{w} = 0 \end{cases}$$

with final condition $\widehat{w}(T,x) = 0$

First order necessary condition:

$$\int_0^T\int_\Omega (u\cdot\widehat{w})\nabla L_g = -\frac{\alpha}{2}\int_0^T\int_\Omega |u-U|^2\nabla L_g - \delta(g-G)$$

Establish the optimality system:

$$\frac{DJ(u(g),g)}{Dg}h = \alpha \int_0^T \int_\Omega L_g(u-U)w + \frac{\alpha}{2}\int_0^T \int_\Omega (\nabla L_g \cdot h)|u-U|^2 + \delta(g-G)h$$
$$= \int_0^T \int_\Omega (\nabla L_g \cdot h)u\widehat{w} + \frac{\alpha}{2}\int_0^T \int_\Omega (\nabla L_g \cdot h)|u-U|^2 + \delta(g-G)h$$

where w is solution to the sensitivity equation

$$\begin{cases} w_t + \nu\Delta w + (u\cdot\nabla)w + (w\cdot\nabla)u + L_g w - (\nabla L_g \cdot h)u = 0 \\ \nabla \cdot w = 0 \end{cases}$$

g is minimizer so $\frac{DJ(u(g))}{Dg}h = 0 \ \forall h \in \mathrm{Tan}\Omega(g)$

$$\int_0^T \int_\Omega (\nabla L_g \cdot h)(u\cdot\widehat{w}) = -\frac{\alpha}{2}\int_0^T \int_\Omega (\nabla L_g \cdot h)|u-U|^2 - \delta(g-G)h \quad \forall h \in \mathrm{Tan}\Omega(g)$$

The semidiscrete-in-time approximation

Formulation

$$\begin{cases} \frac{1}{\Delta t}(u^{(n+1)} - u^{(n)}) - \nu\Delta u^{(n+1)} + u^{(n+1)} \cdot \nabla u^{(n+1)} + \\ \qquad\qquad + \nabla p^{(n+1)} + L_g u^{(n+1)} = f^{(n+1)} \\ \nabla \cdot u^{(n+1)} = 0 \\ u^{(n+1)}(x) = 0, x \in \partial\Omega \\ u^{(0)}(x) = u_0(x) \end{cases}$$

Cost function:

$$\frac{\alpha\Delta t}{2} \sum_{n=1}^{N} L_g \|u^{(n)} - U^{(n)}\|^2 + \frac{\delta}{2}|g - G|^2$$

Theorem (Consistency) Given $\Delta t = \frac{T}{N}$.
For $\Delta t \to 0$, the solution $\{(u^{(n)}, g)\}_{n=1}^{N}$ of the semi-discrete-in-time optimal control problem converges to the solution (u, g) of the corresponding continuous optimal control problem.

- $\{u^N\}$ is uniformly bounded in $\mathbf{L}^2(0, T; V(\Omega))$
- $\{u^N\}$ is uniformly bounded in $\mathbf{L}^\infty(0, T; W(\Omega))$
- $\{u'^N\}$ is uniformly bounded in $\mathbf{L}^2(0, T; V^*(\Omega))$
- Passing the limit

$\mathbf{B}_1 = \mathbf{H}_0^1(\Omega) \times \mathbf{L}_0^2(\Omega) \times \mathbb{R}^2$
$\mathbf{B}_2 = \mathbf{H}^{-1}(\Omega) \times \mathbf{L}_0^2(\Omega)$

$M : \mathbf{B}_1 \to \mathbf{B}_2$ is defined as $M(u^N, p^N, g) = (h^N, z^N)$ if and only if

$$\begin{cases} \frac{1}{\Delta t}(u^{(n+1)} - u^{(n)}, v) + \nu(\nabla u^{(n+1)}, \nabla v) + (u^{(n+1)} \cdot \nabla u^{(n+1)}, v) + \\ \quad +(p^{(n+1)}, \nabla \cdot v) + (L_g u^{(n+1)}, v) = (f^{(n+1)} + h^{(n+1)}, v) \ \forall v \in H_0^1(\Omega) \\ (\nabla \cdot u^{(n+1)}, q) = (z^{(n+1)}, q) \quad \forall q \in L_0^2(\Omega) \\ u^{(n+1)}(x) = 0, x \in \partial\Omega \\ u^{(0)}(x) = u_0(x) \end{cases}$$

$N : \mathbf{B}_1 \to \mathbb{R} \times \mathbf{B}_2$ is defined as follow

$$N(u^N, p^N, g) = \begin{pmatrix} J(u^N, g) - J(\hat{\mathbf{u}}, \widehat{g}) \\ M(u^N, p^N, g) \end{pmatrix}$$

Theorem. Let $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}_0^2(\Omega) \times \mathbb{R}^2$ denote the optimal solution. Then there exists a nonzero Lagrange multiplier $(\psi, \sigma) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$ such that

$$J'(\hat{\mathbf{u}}, \widehat{g}) \cdot (\mathbf{w}, \mathbf{r}, k) + < M'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g}) \cdot (\mathbf{w}, \mathbf{r}, k), (\psi, \sigma) >= 0$$
$$\forall (\mathbf{w}, \mathbf{r}, k) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}_0^2(\Omega) \times \mathbb{R}^2$$

where $< ., . >$ denotes the duality pairing between $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$ and $(\mathbf{H}^1(\Omega))^* \times \mathbf{L}^2(\Omega)$

- $N'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g})$ has a closed range
- $N'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g})$ is not onto

By the Hahn-Banach theorem, there exists $(\xi, \psi, \sigma) \neq 0$ in $R \times \mathbf{B}_2^*$ such that

$$< (\xi, \psi, \sigma), (\beta, \bar{\mathbf{h}}, \bar{\mathbf{z}}) >= 0 \qquad \forall (\beta, \bar{\mathbf{h}}, \bar{\mathbf{z}}) \in \text{Range}(N'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g}))$$

$$\xi \cdot J'(\hat{\mathbf{u}}, \widehat{g}) \cdot (\mathbf{w}, \mathbf{r}, k) + < M'(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g}) \cdot (\mathbf{w}, \mathbf{r}, k), (\psi, \sigma) >= 0 \qquad \forall (\mathbf{w}, \mathbf{r}, k) \in \mathbf{B}_1$$

$$\xi := 1$$

First order necessary condition Let $(\hat{\mathbf{u}}, \hat{\mathbf{p}}, \widehat{g})$ be the optimal solution of the semidiscrete-in-time problem. Then we have

$$\Delta t \sum_{n=1}^{N} \int_{\Omega} (\widehat{u}^{(n)} \cdot \psi^{(n-1)}) \nabla L_{\widehat{g}} dx = -\frac{\alpha}{2} \Delta t \sum_{n=1}^{N} \int_{\Omega} |\widehat{u}^{(n)} - U^{(n)}|^2 \nabla L_{\widehat{g}} dx - \\ - \delta(\widehat{g} - G)$$

where ψ and σ satisfies

$$\begin{cases} -\frac{1}{\Delta t}(\psi^{(n+1)} - \psi^{(n)}, v) + \nu(\nabla \psi^{(n)}, \nabla v) - (\widehat{u}^{(n+1)} \cdot \psi^{(n)}, v) + \\ \quad +((\nabla \widehat{u}^{(n+1)})^T \cdot \psi^{(n)}, v) + (L_{\widehat{g}}^T \psi^{(n)}, v) + (\sigma^{(n)}, \nabla \cdot v) = \\ \quad = \alpha L_{\widehat{g}}(\widehat{u}^{(n+1)} - U^{(n+1)}, v) \\ \qquad \forall v \in H_0^1(\Omega), n = 0, ..., N-1 \\ (\nabla \cdot \psi^{(n)}, q) = 0, \forall q \in L_0^2(\Omega), n = 0, ..., N-1 \\ \psi^{(n)} = 0 \text{ on } \partial\Omega, n = 0, ..., N-1 \\ \psi^{(N)} = 0 \end{cases}$$

$X^h \subset H_0^1(\Omega)$ and $S^h \subset L^2(\Omega)$: finite dimensional subspace

Approximation properties: There exists an integer l and a constant C, independent of h, u and p such that for all $1 \leq k \leq l$ we have

$$\inf_{u_h \in X^h} \|u_h - u\|_1 \leq Ch^k \|u\|_{k+1} \quad \forall u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$$

$$\inf_{p_h \in S^h} \|p - p_h\| \leq Ch^k \|p\|_k \quad \forall p \in H^k(\Omega) \cap L_0^2(\Omega)$$

The LBB condition: There exists a constant C', independent of h such that

$$\inf_{0 \neq q_h \in S^h} \sup_{0 \neq u_h \in X^h} \frac{b(u_h, q_h)}{\|u_h\|_1 \|q_h\|} \geq C' > 0$$

Formulation

$$\begin{cases} \frac{1}{\Delta t}(u_h^{(n+1)} - u_h^{(n)}, v_h) + \nu(\nabla u_h^{(n+1)}, \nabla v_h) + (u_h^{(n+1)} \cdot \nabla u_h^{(n+1)}, v_h) + \\ \quad +(p_h^{(n+1)}, \nabla \cdot v_h) + (L_g u_h^{(n+1)}, v_h) = (f^{(n+1)}, v_h) \quad \forall v_h \in X^h(\Omega) \\ \\ (\nabla \cdot u_h^{(n+1)}, q_h) = 0 \quad \forall q_h \in S_0^h(\Omega) \end{cases}$$

Cost function:

$$\frac{\alpha \Delta t}{2} \sum_{n=1}^{N} L_g \| u_h^{(n)} - U^{(n)} \|^2 + \frac{\delta}{2} |g - G|^2$$

Problem. Given $\Delta t = T/N$, find $(\mathbf{u}_h, \mathbf{p}_h, g)$ in $\mathbf{X}^h(\Omega) \times \mathbf{S}_0^h(\Omega) \times R^2$ such that the state equation is satisfied and the cost function is minimized.

The adjoint equation

$$\begin{cases} -\frac{1}{\Delta t}(\psi_h^{(n+1)} - \psi_h^{(n)}, v_h) + \nu(\nabla\psi_h^{(n)}, \nabla v_h) - (\widehat{u_h}^{(n+1)} \cdot \nabla\psi_h^{(n)}, v_h) + \\ \quad +((\nabla\widehat{u}_h^{(n+1)})^T \cdot \psi_h^{(n)}, v_h) + (L_{\widehat{g}}^T \psi_h^{(n)}, v_h) + (\sigma_h^{(n)}, \nabla \cdot v_h) = \\ \quad = \alpha L_{\widehat{g}}(\widehat{u}_h^{(n+1)} - U^{(n+1)}, v_h) \quad \forall v_h \in X^h(\Omega) \\ (\nabla \cdot \psi_h^{(n)}, q_h) = 0, \forall q_h \in S_0^h(\Omega) \\ \psi_h^{(n)} = 0 \text{ on } \partial\Omega \\ \psi_h^{(N)} = 0 \end{cases}$$

Future work

- Computational experiment
- Analysis the optimal control problem, provided that there are more than one porous medium in the domain

Reference

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Thank you for your attention!