Optimal filtering of infinite dimensional systems with stationary and mobile sensor networks

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Interdisciplinary **C**enter for **A**pplied **M**athematics
Virginia Tech

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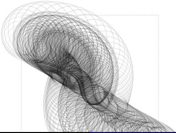

Two dimensional parabolic problem

Consider the **convection-diffusion** process in $\Omega = (0,1) \times (0,1)$ and in $t \in (0, \tau)$

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with ξ a Gaussian random variable. The natural state space for the problem is $\mathscr{H} = L^2(\Omega)$ and the domain of the differential operator in the right hand side is $H^2(\Omega) \cap H_0^1(\Omega)$.

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$$T(0,x,y) = T_0(x) + \xi$$

is a stochastic process with values in $\mathscr{H} = L^2(\Omega)$.

Bensoussan, A. *Filtrage Optimal des Systèmes Lin'eaires* (Dunod, 1971)

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Stationary Networks The position of the sensor remains constant. The design variables are the positions of the sensors in the domain.

Mobile Networks The positions of the sensors are described by controlled differential equations and their initial positions are **fixed**. The design variable are the the controls.

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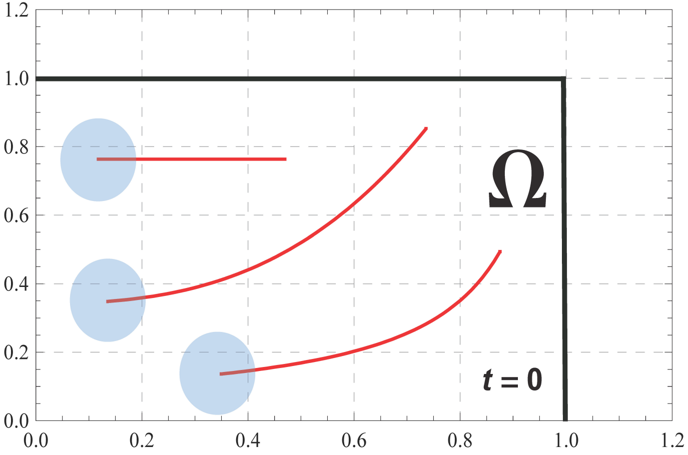
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1.0
0.8
0.6
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0.2
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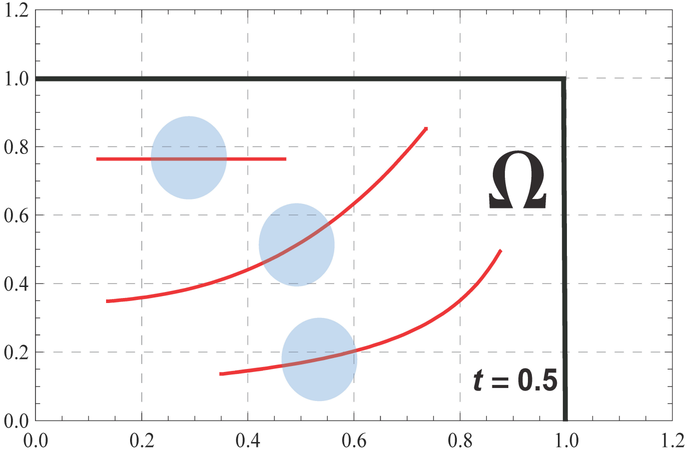
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Ω
t = 0


1.2
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0.8
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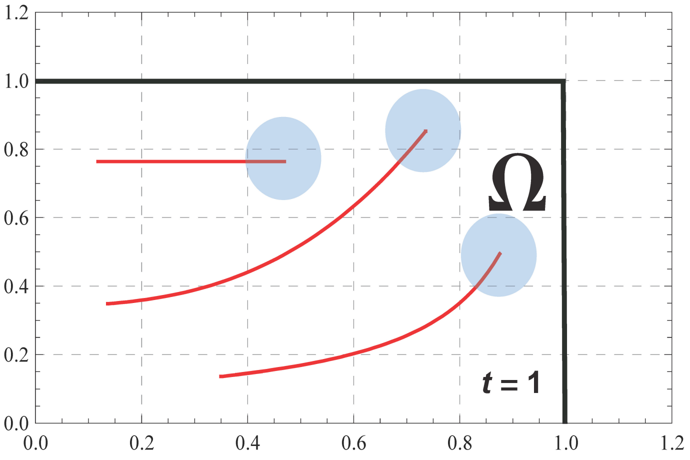
1.0
1.2
Ω
t = 0.5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Then, we can write this as

$$h(t) = C_u(t)\, T(t, \cdot) + \nu(t),$$

and for each $t \in [0, \tau]$, the operator $C_u^* C_u(t)$ is of trace class.

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Question: How are we going to choose $C_u(t)$?

If we construct a Kalman filter, then the covariance operator $\Sigma(t)$ between the real state $z(t)$ and the estimated one $\hat{z}(t)$ is the mild solution of the Riccati differential equation

$$\dot{\Sigma} = A\Sigma + \Sigma A + BR_1 B^* - \Sigma C_u^* R_2^{-1} C_u \Sigma,$$

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but when is Σ_u a trace class operator?

The Riccati integral equation

We are interested in trace-class valued solutions of

$$\Sigma(t) = T(t)\Sigma_0 T^*(t) + \int_0^t T(t-s)(BB^* - \Sigma(C^*C)\Sigma)(s)T^*(t-s)\mathrm{d}s,$$

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where the integral is well defined as a Bochner integral, so the solution is a "uniform" one.
This is not the usual case, for example if $T(t)$ is semigroup of linear operators with generator A,

$$\int_0^1 T(t)\,\mathrm{d}t,$$

is a well defined Bochner integral IF AND ONLY IF A is bounded!

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Then there is a unique solution $t \mapsto \Sigma(t)$ of the integral Riccati equation which belongs to $L^2([0,\tau], \mathscr{I}_2)$ and even more the same solution belongs to $\mathscr{C}([0,\tau], \mathscr{I}_1)$.

Importance of the previous Theorem

a) We have general conditions over $B(\cdot)$ and $C(\cdot)$ for which $\Sigma(\cdot)$ is Trace-Class-valued.

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c) The integral is a well-defined Bochner one (Approximation is possible through discretization of $[0,\tau]$)

Mobile Sensors

Let $J : \mathcal{U} \mapsto \mathbb{R}$ be defined as

$$J(u) = \int_0^\tau \mathrm{Tr}(Q(t)\Sigma_u(t))\,\mathrm{d}t,$$

with $Q(\cdot) \in L^\infty([0,\tau]; \mathscr{L}(\mathscr{H}))$ and $Q(t) \geq 0$.

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$$J(\tilde{u}) = \inf_{u \in \mathcal{U}} J(u).$$

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Let $J : \Omega \mapsto \mathbb{R}$ be defined as

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one dimensional convection-diffusion process

$$T_t = \epsilon T_{xx} + a_x T_x + b(x, r, a)\eta(t),$$

on $0 \leq t \leq 1$, and $0 \leq x \leq 1$. With $T(t, 0) = T(t, 1) = 0$ and $T(0, x) = T_0(x)$

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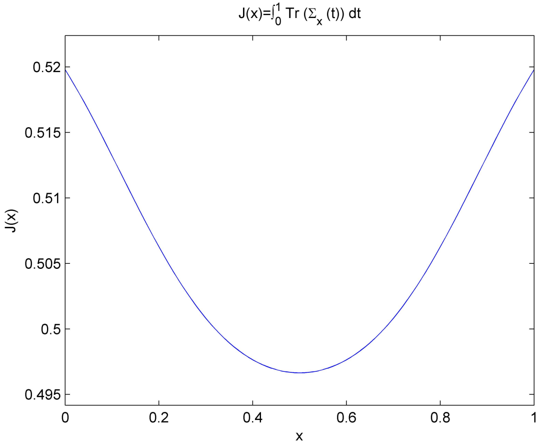
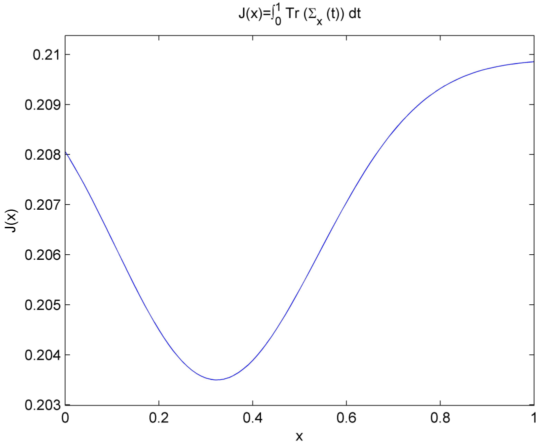
In this example

$$b(x, r, a) = e^{-r(x-a)^2} \qquad c(x-y) = e^{-10(x-y)^2}$$

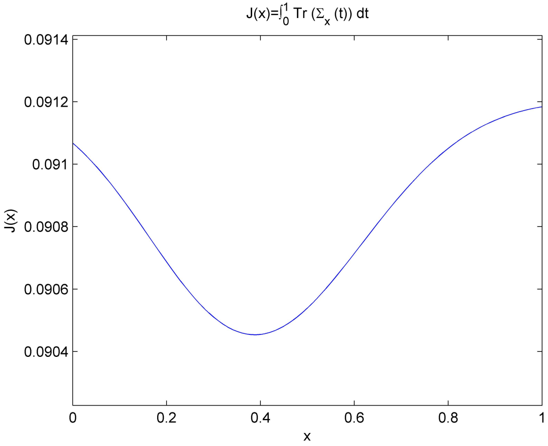
$\epsilon = a_x = 0$ and $b(x, 0, a) = 1$



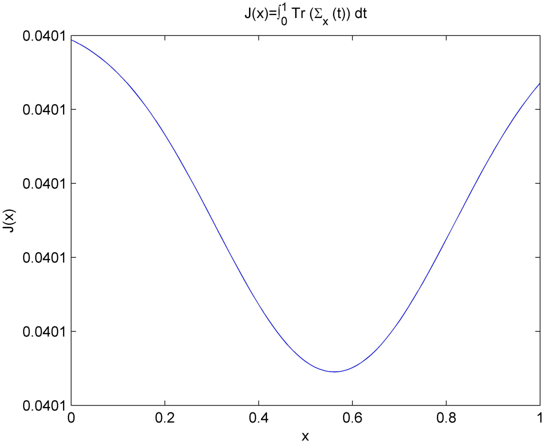
$\epsilon = a_x = 0$ and $b(x, 10, 0.3) = e^{-10(x-0.3)^2}$



$\epsilon = 0.1,\ a_x = 0$ and $b(x, 10, 0.3) = e^{-10(x-0.3)^2}$



$\epsilon = 0.1,\ a_x = 5$ and $b(x, 10, 0.3) = e^{-10(x-0.3)^2}$



Consider

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on $0 \leq t \leq 1$, and $\mathbf{x} = (x,y) \in \Omega \equiv (0,1) \times (0,1)$. With $T(t,\mathbf{x})\Big|_{x \in \partial\Omega} = 0$ and $T(0,x) = T_0(x)$.

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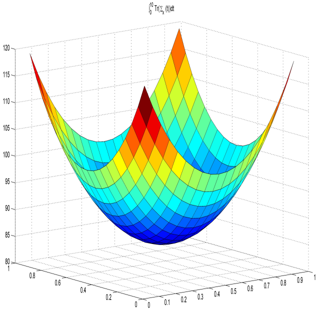
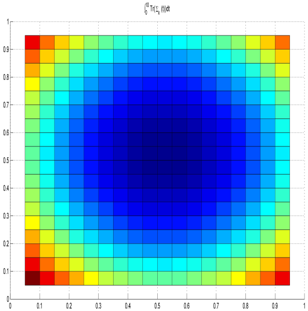
In this example

$$b(x, y, r) = 10 + 40 e^{-r\left((x-0.1)^2 + (y-0.1)^2\right)}$$
$$c(\mathbf{x} - \mathbf{y}) = e^{-20\left((x_1-y_1)^2 + (x_2-y_2)^2\right)}$$

$\epsilon = a_x = a_y = 0$ and $b(x, y, 0) = 50$



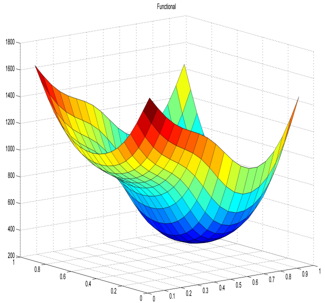
(a) Side View



(b) Top View

Figure: Value of $J(\mathbf{x}) = \int_0^{10} \mathrm{Tr}(\Sigma_{\mathbf{x}}(t))\, \mathrm{d}t$ where $\mathbf{x}$ is the position of the sensor

$\epsilon = 0.01$, $a_x = 5$, $a_y = 0$ and $b(x, y, 0) = 50$



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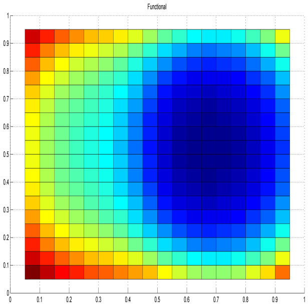

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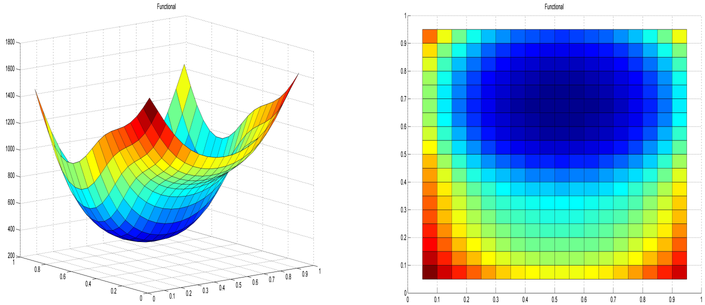

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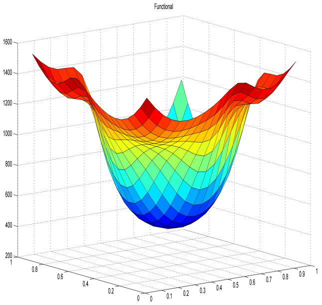
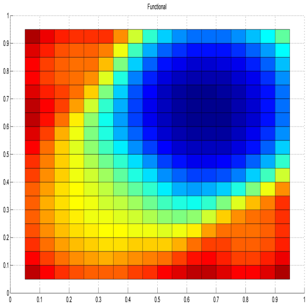
$\epsilon = 0.01$, $a_x = 5$, $a_y = 5$ and $b(x, y, 0) = 10$



(a) Side View



(b) Top View

Figure: Value of $J(\mathbf{x}) = \int_0^{10} \mathrm{Tr}(\Sigma_{\mathbf{x}}(t))\,\mathrm{d}t$ where $\mathbf{x}$ is the position of the sensor

$\epsilon = 0.01$, $a_x = a_y = 0$ and $b(x, y, 5)$

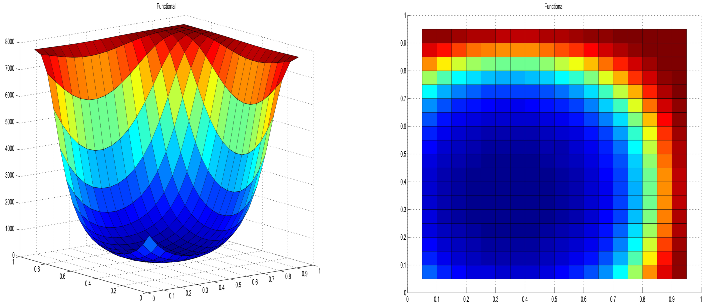


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Figure: Value of $J(\mathbf{x}) = \int_0^{10} \mathrm{Tr}(\Sigma_{\mathbf{x}}(t))\, \mathrm{d}t$ where $\mathbf{x}$ is the position of the sensor

Consider the one dimensional convection diffusion

$$T_t = \epsilon T_{xx} - aT_x + b(x)\eta(t),$$

on $0 \leq t \leq 0.2$, and $0 \leq x \leq 1$. With $T_x(t,0) = T_x(t,1) = 0$ and $T(0,x) = T_0(x)$.

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Suppose that the family of sensors $\mathcal{F}$, correspond to those which move uniformly in time, from $x_0 \in [0, 1]$ to $x_1 \in [0, 1]$ and with range $\delta = 0.05$.

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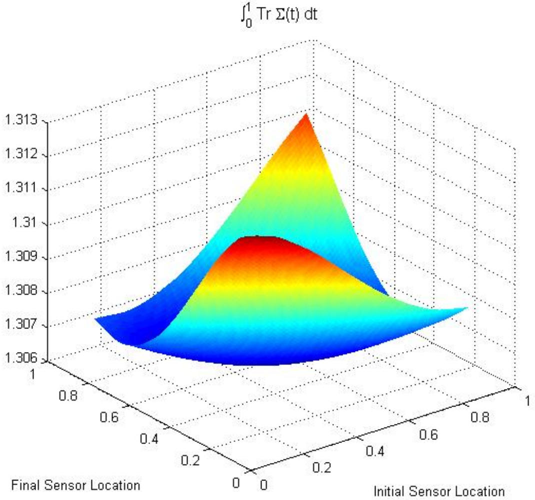
Then, we can parameterize $J(C) = \int_0^1 \mathrm{Tr}(\Sigma)\,\mathrm{d}t$, with x_0 and x_1 as $J(x_0, x_1)$...

Finite element approximation with $n = 128$



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Apparently the minimum is attained when $x_0 \simeq 0.592$ and $x_1 \simeq 0.590...$ which is more or less stationary.

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is Frechet differentiable as a mapping $J : L^2([0,1]) \mapsto \mathbb{R}$, and then...let's try to apply Steepest Descent to this unconstrained minimization problem and see what happens!

The sensors

Assume we have 3 sensors located at the points $(0.6, 0.4), (0.5, 0.5)$ and $(0.4, 0.6)$ and their trajectories are given by the equations

$$\vec{x}_i(t,u) = \begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{b}u_i(s)\,\mathrm{d}s$$

where

$$\mathbf{A} = \begin{pmatrix} -1 & 0.3 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix}.$$

Steepest Descent Method

We will use a gradient descent method to try to compute a local minimizer of the problem.

1. Start with the control with some choice $u^0(t) = (u_1^0(t), u_2^0(t), u_2^0(t))$.

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1. Start with the control with some choice $u^0(t) = (u_1^0(t), u_2^0(t), u_2^0(t))$.
2. Update the control as

$$u^{n+1}(t) = u^n(t) - \alpha_n J'(u^n)(t),$$

where $J'(u)$ is the gradient of J at u and α_n is chosen if possible as

$$\alpha_n = \arg\min_\alpha J(u^n - \alpha J'(u^n)),$$

and stop if $J(u^{n+1})$ is not decreased by at least 2% with respect to $J(u^n)$.

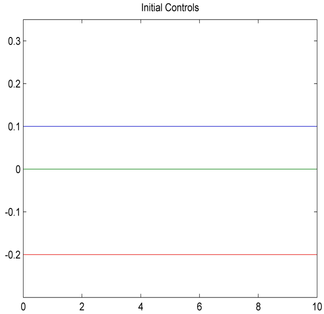
Consider

$$\frac{\partial}{\partial t} T = 0.01 \Delta T + b(x, y, a)\eta(t);$$

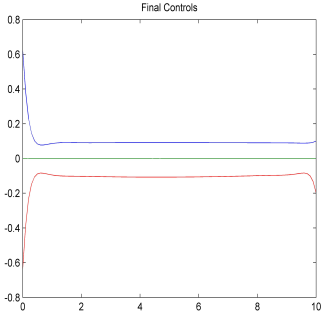
on $0 \leq t \leq 1$, and $\mathbf{x} = (x, y) \in \Omega \equiv (0,1) \times (0,1)$. With $T(t, \mathbf{x})\Big|_{x \in \partial\Omega} = 0$ and $T(0, x) = T_0(x)$.
In this example

$$b(x, y, a) = 10 + ae^{-5\left((x-0.1)^2 + (y-0.9)^2\right)}$$

$b(x, y, 0) = 10$



(a) Initial Controls



(b) Final Controls

Figure: Initial and Final controls (16 iterations and approximately 1 hour)

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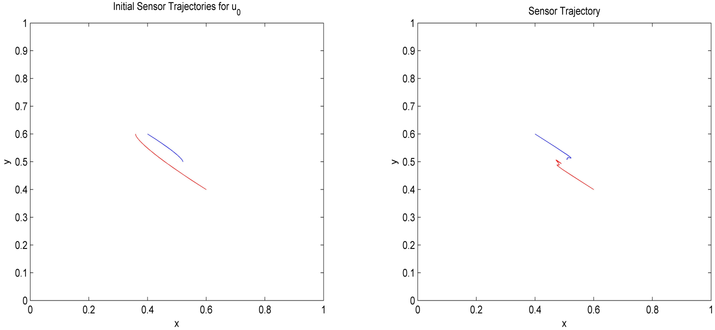
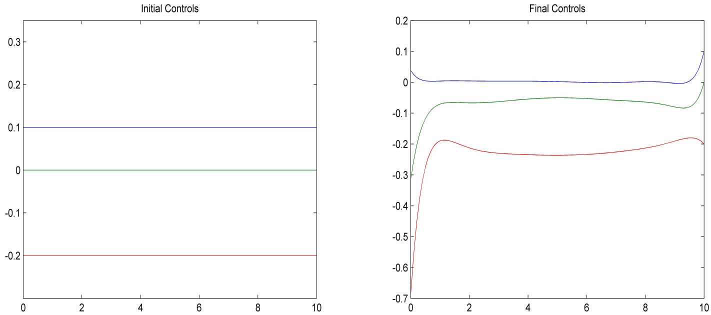


(a) Initial Trajectories (b) Final Trajectories

Figure: Initial and Final Iterations(16 iterations and approximately 1 hour)

$$b(x, y, 10) = 10 + 10e^{-5\left((x-0.1)^2+(y-0.9)^2\right)}$$



(a) Initial Controls

(b) Final Controls

Figure: Initial and Final controls (12 iterations and approximately 45 minutes)

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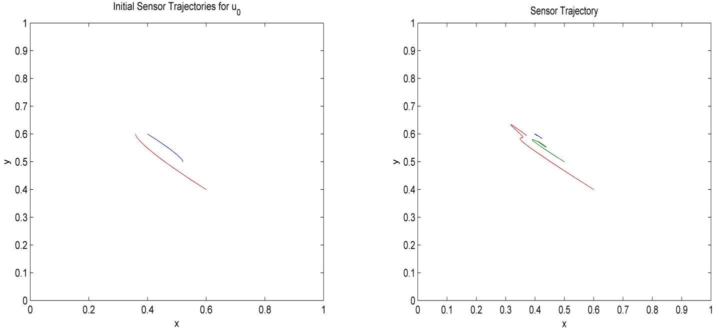

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THANK YOU!