Parameter estimation

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Problem formulation

Cost function and constraints

Find $q_0 \in Q^{ad}$ solution of

$$\frac{1}{2} \min_{q \in Q^{ad}} \|z(q) - z_{data}\|_Z^2 + \frac{\beta}{2}\|q\|_{Q^{ad}}^2, \beta > 0$$

subject to the constraints

$$(\mathcal{P})_q \begin{cases} \frac{d}{dt} z(t;q) = A(q)z(t;q) + f(q,t,z(t;q)) & z(t) \in H^1(\Omega), \\ \Omega \subset R^n \text{ with compact closure;} & \\ z(0) = z_0 & t \in [0,T]. \end{cases}$$

$$A(q)z = -\sum_{i,j=1}^{n} (a_{i,j}(x) z_{x_i})_{x_j} - b \cdot \nabla z$$

$z(q) \in K = \{u \geq 0 : u \in Z = L^2(0,T;H^1(\Omega))\}$

Assumptions on A and construction of Q^{ad}

Operator assumptions

- $a_{i,j}(x) \in C^1(\overline{\Omega})$, , $a_{i,j} = a_{j,i}, i,j = 1,...,n$
- $\rho \geq a_{i,j}(x) \geq \gamma > 0 \; i,j = 1,...,n$
- $\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \omega||\xi||^2 \; \forall \xi \in \mathbb{R}^n$
- $b \in C^2(\overline{\Omega}) \; \rho \geq b \geq \gamma > 0$
- A is the infinitesimal generator of an operator T such as
- $\exists \epsilon_0 > 0 \; such \; \forall q \in Q^{ad} \; \exists C_q > 0 ||T(t;q)|| < C_q e^{-\epsilon_0 t}$
- *The mapping* $q \to D_q A(q) T(.; q_0)$ *from* Q^{ad} *into* $L^1(0,\infty; H^1(\Omega))$*is Lipschitz continuous at* $q_o, \forall q_0 \in Q^{ad}$

Admissible set

- $Q^{ad} = \{ q = (a,b,c)$ *as above and c free element on* f , $c \in C^1(\overline{\Omega}), \rho \geq c \geq \gamma > 0 \; \}$

Assumptions

Assumptions on f

- *f is locally Lipschitz continuous on t and z* $|f(q,t_1,z_1) - f(q,t_2,z_2)| \leq L(q)(|t_1 - t_2| + ||z_1 - z_2||_Z)$
- *The mapping* $q \rightarrow f(q,.;z)$ *from* Q^{ad} *into* $L^\infty(0,T;H^1(\Omega))$ *is Lipschitz continuous with constant independent of z*

Under the previous assumptions

- A is generator of an Analytic semigroup over $H^1(\Omega)$ by Pazi
- $z(t,q)$ *is* $q-$ *Frechet differentiable* by Herdman and Spies
- *There is a unique solution* q_0 *to* $(\mathcal{P})_q$ *by Banks and Kunisch*

Necessary conditions for optimality at q_0

Reformulation

- $F(q) = \frac{1}{2}\|z(q) - z_{data}\|_Z^2 + \frac{\beta}{2}\|q\|_{Q^{ad}}^2$
- $g(q) = z(q)$

Lagrangian

$\exists \lambda^* \in K^+ = \{\lambda^* : \lambda^* \in Z^*, \lambda^*(u) \geq 0 \ \forall u \in K\}$ *such as* :

- $\lambda^*(g(q_0)) = 0$
- $L(q) = F(q) - \lambda^* g(q)$
- $< L_q(q_0), h >= (z_q - z_{data}) \cdot v_h(t) + \beta q \cdot v_h(t) - \lambda^*(v_h(t)) = 0$ $\forall h \in Q^{ad}$

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Frechet derivative and sensitivity equation

Frechet derivative

The $q-$ Frechet derivative of $z(t;q)$ evaluated at q_0 applied to h i.e $[D_q z(t;q_0)]h$ is $v_h(t)$ solution of :

Sensitivity equation

$$(\mathcal{S})_q \begin{cases} \frac{d}{dt} v_h(t) = (A(q_0) + f_z(q_0, t, z(t;q_0)))v_h(t) \\ + f_q(q_0, t, z(t;q_0)))h + D_q A(q) T(t;q_0) z_0|_{q=q_0} h \\ + \int_0^t D_q A(q) T(t-s;q_0)|_{q=q_0} h f(q_0, s, z(s;q_0)))ds \\ v_h(0) = 0. \end{cases}$$

Example for numerical testing

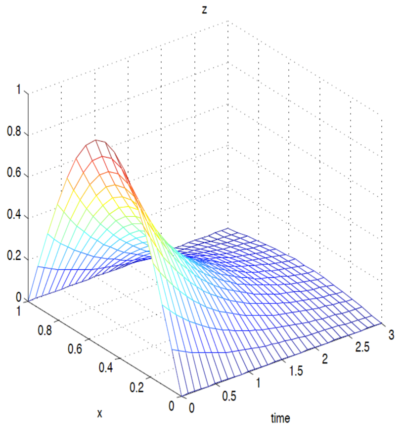
State equation

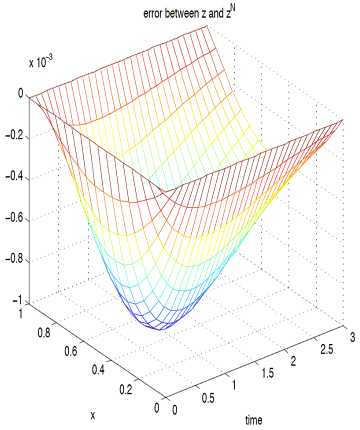
$$(\mathcal{A})_q \begin{cases} \frac{d}{dt} z(t) = -\nabla \cdot (q \nabla z(t)), & z(t) \in H^1((0,1)); \\ z(0) = sin(\pi x), & t \in [0,1]. \end{cases}$$

Sensitivity equation

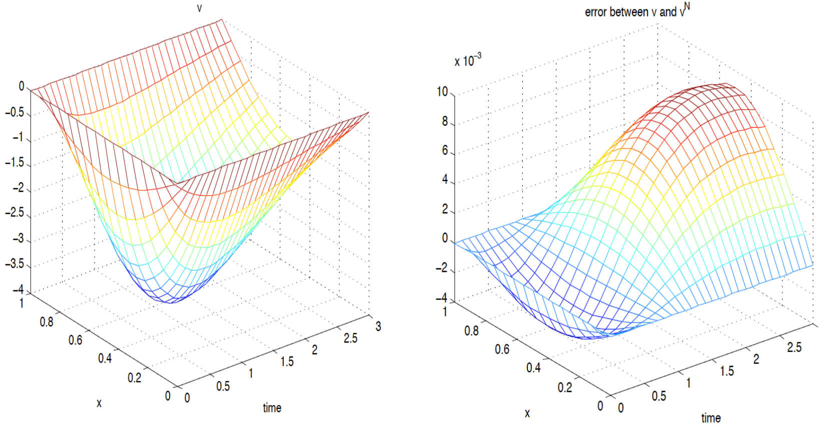
$$(\mathcal{S})_q \begin{cases} \frac{d}{dt} v_h(t) = -\nabla \cdot (q_0 \nabla v_h(t)) - \nabla \cdot (h \nabla z(t)); \\ \\ v_h(0) = 0. \end{cases}$$

Numerical results on z q_0=0.01 h=1





Numerical results on v_h q_0=0.01 h=1



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Future research topics

Evaluate the Frechet derivatives of the sensitivity equation

- *Evaluate $z(q)$ through finite elements (vector)*
- *Evaluate $v_h(q)$ through time dependent matrix, since the Frechet derivative is a linear operator*
- *Storage the matrices using SVD and study its effect*