The Discontinuous Galerkin Method for Hyperbolic Problems on tetrahedral meshes: A posteriori Error Estimation

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5. Conclusion: Summarize results and described future work.

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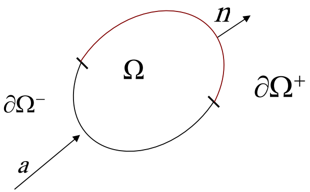
  - Has a simple communication pattern between elements with a common face that makes it useful for parallel computation.
  - Can handle problems with complex geometries to high order.
  - Does not require continuity across element boundaries
- A posteriori Error Estimation
  - $u - U_h \approx E$
  - Asymptotic behavior of the error
  - Drive Adaptive refinement

DG formulation and preliminary results

A model problem

$$\begin{cases} a.\nabla u = f(x,y,z), & (x,y,z) \in \Omega = [0,1]^3 \\ u|_{\partial\Omega} = g(x,y,z) \end{cases} \tag{1}$$



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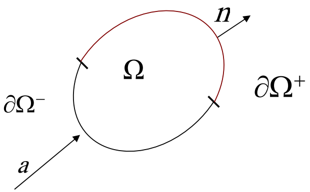
where
f and g are selected such that the exact solution $u \in C^\infty(\Omega)$.
a denote a constant non zero velocity vector.
$\partial\Omega = \partial\Omega^+ \cup \partial\Omega^- \cup \partial\Omega^0$,



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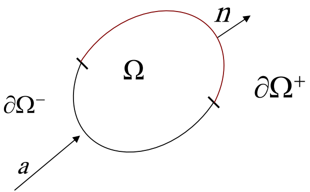

$$\begin{aligned} \partial\Omega^- &= \{(x,y,z) \in \partial\Omega |\, a.n < 0\}, \text{is the inflow boundary,} \\ \partial\Omega^+ &= \{(x,y,z) \in \partial\Omega |\, a.n > 0\}, \text{is the outflow boundary and} \\ \partial\Omega_0 &= \{(x,y,z) \in \partial\Omega |\, a.n = 0\}, \text{is the characteristic boundary.} \end{aligned}$$

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we partition the domain Ω into a regular mesh having N tetrahedra elements Δ_j, $j = 1, ..., N$ of diameter $h > 0$, for simplicity We refer to an arbitrary element by Δ.
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Class and Types of elements

	Type I	*Type II*	*Type III*
Class I	T_{11}	T_{12}	T_{13}
Class II	T_{21}	T_{22}	
Class III	T_{31}		

DG formulation and preliminary results

Some examples for Class and Types of Elements

T_{11}

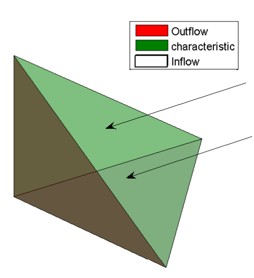

Figure: T_{11}: One Outflow, One Inflow and Two Characteristics

T_{12}

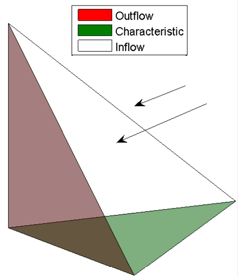

Figure: T_{12}: One Outflow, Two Inflow and One Characteristics

DG formulation and preliminary results

$\mathcal{L}^2$ Orthogonal basis functions

$$\varphi^{p}_{q,r}(\xi,\eta,\zeta) = \overline{P}^{0,0}_{p}(\xi,\eta,\zeta)\,\overline{P}^{2p+1,0}_{q}(\eta,\zeta)\,\overline{P}^{2p+2q+2,0}_{r}(\zeta),$$

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where

$$\begin{aligned}
\overline{P}^{0,0}_{p}(\xi,\eta,\zeta) &= (1-\zeta-\eta)^{p}\,\hat{P}^{0,0}_{p}\left(\frac{2\xi}{1-\eta-\zeta}-1\right),\\
\overline{P}^{2p+1,0}_{q}(\eta,\zeta) &= (1-\zeta)^{q}\,\hat{P}^{2p+1,0}_{q}\left(\frac{2\eta}{(1-\zeta)}-1\right),\\
\overline{P}^{2p+2q+2,0}_{r}(\zeta) &= \hat{P}^{2p+2q+2,0}_{r}(2\zeta-1).
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\end{aligned}$$

$$P_{n}^{\alpha,\beta}(x) = \frac{(-1)^{n}}{2^{n}n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^{n}}{dx^{n}}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right],\ \alpha,\beta > -1.$$

Satisfy $\mathcal{L}^2$ orthogonality and

$$\varphi_{q,r}^{p}(\xi,\eta,\zeta)=\overline{P}_{p}^{0,0}(\xi,\eta,\zeta)\,\overline{P}_{q}^{2p+1,0}(\eta,\zeta)\,\overline{P}_{r}^{2p+2q+2,0}(\zeta),$$

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Satisfy $\mathcal{L}^2$ orthogonality and is complete in the space $\mathcal{P}_p$

$$\int_0^1\int_0^{1-\eta}\int_0^{1-\eta-\zeta}\varphi_{ij}^{m}\varphi_{kl}^{n}\,d\zeta\, d\eta\, d\xi = c_{ij,kl}^{mn}\delta_{ik}\delta_{jl}\delta_{mn},$$

Multiply (1) by a test function v, integrate over an Δ, apply Stokes' theorem:

$$\begin{aligned} &\int\int_{\Gamma^-} a.nuvd\sigma + \int\int_{\Gamma^+} a.nuvd\sigma + \int\int\int_{\Delta} (-a.\nabla v)\, udxdydz \\ = &\int\int\int_{\Delta} fvdxdydz, \end{aligned} \tag{2}$$

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$$\int\int_{\Gamma^-} a.nU^- Vd\sigma + \int\int_{\Gamma^+} a.nUVd\sigma + \int\int\int_{\Delta} (-a.\nabla V)\, Udxdydz = \int\int\int_{\Delta} fVdxdydz, \text{ for all } V \in \mathcal{P}_p \quad (3)$$

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here $U^- \approx u$ in Γ^-. Let Δ such that $\Gamma^- \subset \partial\Omega^-$, and subtract (3) from (2) with $v = V$ to obtain the DG orthogonality conditions for the local error $\epsilon = u - U$

$$\int\int_{\Gamma^-} a.n\epsilon^- V d\sigma + \int\int_{\Gamma^+} a.n\epsilon V d\sigma + \int\int\int_{\Delta} (-a.\nabla V)\, \epsilon dxdydz = 0, \quad (4)$$

for all $V \in \mathcal{P}_p$. By the standard affine mapping

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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} V_2 - V_1 & V_3 - V_1 & V_4 - V_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} + V_1$$

map Δ with vertices $V_i = (x_i, y_i, z_i)$, $i = 1, 2, 3, 4$ into the canonical tetrahedron $\hat{\Delta}$ with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.
Then the local error satisfy these orthogonality conditions on the canonical element

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$$\int\int_{\hat{\Gamma}^-} \hat{a}.\hat{n}\hat{\epsilon}^- V d\hat{\sigma} + \int\int_{\hat{\Gamma}^+} \hat{a}.\hat{n}\hat{\epsilon}\hat{V} d\sigma + \int\int\int_{\hat{\Delta}} \left(-\hat{a}.\nabla\hat{V}\right)\hat{\epsilon} d\xi d\eta d\zeta = 0, \qquad (5)$$

for all $\hat{V} \in \mathcal{P}_p$.

If u is analytic, we can write the local error as a Maclaurin series

$$\epsilon(\xi,\eta,\zeta) = \sum_{k=0}^{\infty} Q_k(\xi,\eta,\zeta)\, h^k, \text{ where } Q_k \in \mathcal{P}_k \tag{6}$$

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Lemma

If $Q_k \in \mathcal{P}_k,\ k = 0, 1, ..., p$ satisfies

$$\int\int_{\Gamma^+} a.nQ_k V d\sigma + \int\int\int_{\Delta} (-a.\nabla V)\, Q_k d\xi d\eta d\zeta = 0,\ \forall V \in \mathcal{P}_p. \tag{7}$$

Then $Q_k = 0,\ 0 \le k \le p$.

Then we obtain the following expression for the local error.

Theorem

Let $u \in \mathcal{C}^{\infty}(\Delta)$ and $U \in \mathcal{P}_p(\Delta)$ be the solutions of (1), then the local finite element error can be written as

$$\epsilon(\xi, \eta, \zeta) = \sum_{k=p+1}^{\infty} h^k Q_k(\xi, \eta, \zeta), \tag{8}$$

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$$
\begin{aligned}
u - U \quad &\approx \quad E = Q_{p+1} h^{p+1} = \sum_{i=0}^{p+1} \sum_{j=0}^{p+1} \sum_{k=0}^{p+1} c_{j,k}^{i} \varphi_{j,k}^{i} \\
&= \quad \sum_{i=0}^{p+1} \sum_{j=0}^{p+1} C_{i-j,j}^{p+1} \chi_{i,j}^{p+1}
\end{aligned}
$$

where we have the degree of freedom

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$$= \sum_{i=0}^{p+1} \sum_{j=0}^{p+1} C_{i-j,j}^{p+1} \chi_{i,j}^{p+1}$$

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$$\dim \left\{ \varphi_{j,k}^{i},\, 0 \le i,j,k \le p+1 \right\} = \dim \mathcal{P}_{p+1}$$
$$= \frac{(p+2)(p+3)(p+4)}{6} = O\left(p^3\right)$$

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$$\dim \left\{ \varphi^i_{j,k},\ 0 \leq i, j, k \leq p+1 \right\} = \dim \mathcal{P}_{p+1} = \frac{(p+2)(p+3)(p+4)}{6} = O\left(p^3\right)$$

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$$\dim \left\{ \chi^{p+1}_{i,j},\ 0 \leq i, j \leq p+1 \right\} = \frac{(p+2)(p+3)}{2} = O\left(p^2\right)$$

The leading term E satisfy these orthogonality conditions

$$\int\int_{\Gamma^-} a.nE^- V d\sigma + \int\int_{\Gamma^+} a.nEV d\sigma + \int\int\int_{\Delta_e} (a\cdot\nabla E)\, V dxdydz \; = 0, \; \forall V \in \mathcal{P}_p$$

where $E^- = u - U^-$, we choose $U|_{\Gamma^-} = U^- = u$, then E^-=0, after mapping to the Canonical element we get

$$\int\int_{\hat{\Gamma}^+} \hat{a}.\hat{n}\hat{E} V d\sigma + \int\int\int_{\hat{\Delta}} \left(\hat{a}\cdot\nabla\hat{E}\right) V d\xi d\eta d\zeta = 0, \; \forall V \in \mathcal{P}_p$$

Let $\lambda = \frac{\alpha}{\beta}$, $\mu = \frac{\gamma}{\beta}$ where $(\alpha, \beta, \gamma) = \hat{a}$.

Then the function $\chi_{i,j}^{p+1}$ computed on the reference tetrahedra for each class of elements, using Mathematica, and are given in terms of $\varphi_{j,k}^{i}$ as:

Basis for the leading term of the local discretization error

▪ Example of Basis functions for element of Class I, $p = 0, 1$

Then the function $\chi_{i,j}^{p+1}$ computed on the reference tetrahedra for each class of elements, using Mathematica, and are given in terms of $\varphi_{j,k}^{i}$ as:

- *Class I* (where we have one outflow)

$p = 0$	$\chi_{0,0}^{1} = \varphi_{0,0}^{1}$ $\chi_{1,0}^{0} = \frac{2}{3}\varphi_{0,0}^{0} + \varphi_{1,0}^{0}$ $\chi_{0,1}^{0} = -\frac{1}{3}\varphi_{0,0}^{0} + \varphi_{0,1}^{0}$
$p = 1$	$\chi_{0,0}^{2} = \varphi_{0,0}^{2}$ $\chi_{1,0}^{1} = \frac{4}{5}\varphi_{0,0}^{1} + \varphi_{1,0}^{1}$ $\chi_{0,1}^{1} = -\frac{1}{5}\varphi_{0,0}^{1} + \varphi_{0,1}^{1}$ $\chi_{2,0}^{0} = \frac{1}{10}\varphi_{0,1}^{0} + \frac{4}{5}\varphi_{1,0}^{0} + \varphi_{2,0}^{0}$ $\chi_{1,1}^{0} = \frac{3}{5}\varphi_{0,1}^{0} - \frac{1}{5}\varphi_{1,0}^{0} + \varphi_{1,1}^{0}$ $\chi_{0,2}^{0} = -\frac{1}{2}\varphi_{0,1}^{0} + \varphi_{0,2}^{0}$

Table 1: Basis functions for element of *Class I*

Basis for the leading term of the local discretization error

- *Class II* (where we have two outflow)

$p = 0$	$\chi^{1}_{0,0} = \varphi^{1}_{0,0} + \frac{\lambda}{3\lambda+3}\varphi^{0}_{0,0}$ $\chi^{0}_{1,0} = \varphi^{0}_{1,0} + \frac{-\lambda+2}{3\lambda+3}\varphi^{0}_{0,0}$ $\chi^{0}_{0,1} = -\frac{1}{3}\varphi^{0}_{0,0} + \varphi^{0}_{0,1}$

Table 2: Basis functions for element of *Class II*

- *Class III* (where we have three outflow)

$p = 0$	$\chi^{1}_{0,0} = \varphi^{1}_{0,0} + \frac{\lambda}{3\lambda+3\mu+3}\varphi^{0}_{0,0}$ $\chi^{0}_{1,0} = \varphi^{0}_{1,0} - \frac{(\lambda-2)}{3\lambda+3\mu+3}\varphi^{0}_{0,0}$ $\chi^{0}_{0,1} = \varphi^{0}_{0,1} - \frac{\lambda-3\mu+1}{3\lambda+3\mu+3}\varphi^{0}_{0,0}$

Table 3: Basis functions for element of *Class III*

The DG solution U_e satisfy on the physical elementts Ω_e

$$\int\int_{\Gamma^-} a.n\left(\tilde{U}^- - U\right) Vd\sigma + \int\int\int_{\Omega_e} (a\cdot\nabla U)\, Vdxdydz = \int\int\int_{\Omega_e} fVdxdydz.$$

and the leading term E satisfy on

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In this analysis we use the local and global effectivity indices in the $\mathcal{L}^2$ norm

$$\theta_e = \frac{\|E\|_{\mathcal{L}^2(\Omega_e)}}{\|e\|_{\mathcal{L}^2(\Omega_e)}}, \text{ and } \theta = \frac{\|E\|_{\mathcal{L}^2(\Omega)}}{\|e\|_{\mathcal{L}^2(\Omega)}}$$

Under mesh refinement, the effectivity indices should approach unity.

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7. go to (3)

Remark

i) Structured meshes: This meshes obtained by partitioning the domain into n^3 cube for $n = 1$, 2,3, 4, 5, 6, 7,8 and dividing each cube into five tetrahedrons (where $h_{\max} = \frac{\sqrt{2}}{n}$). Thus, the meshes have $N = 5 \times n^3 = 40$, 135, 320, 625, 1080, 1715 and 2560 tetrahedra elements.
ii) Unstructured meshes: These meshes obtained by COMSOL software for $h_{\max} = \dfrac{1}{n}$ (for $n = 1, 2, 3, 4, 5, 6, 7, 8$) with number of elements $N = 24, 192$, 476, 943, 2121, 3731, 5846 and 8713.

Computational Examples

Structured meshes

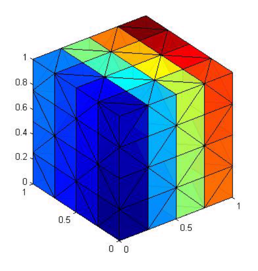

Figure: Tetrahedral mesh with $N = 5 \times 8^2 = 320$ elements

Unstructured meshes

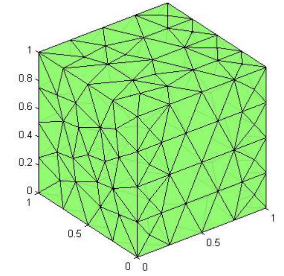

Figure: TTetrahedral meshes obtained by COMSOL with $N = 953$ elements

Example 1: We consider the following linear hyperbolic problem

$$-3u_x - 7u_y + 13u_z = 3e^{x+y+z}, \; (x, y, z) \in \Omega = [0, 1]^3,$$

and select the initial and boundary conditions such that the exact solution is

$$u(x, y, z) = e^{x+y+z}.$$

We solve this problem using $\mathcal{P}_p$, with the exact boundary condition i.e. $U^- = u$, on the first meshes and compar the two methods for $p = 0, 1, 2, 3$.

Computational Examples

- Results of Problem 1 in structured meshes, $p = 0$

$\mathcal{P}_p$ (**Example 1**)						
With correction						
p	N	$\|u - U\|_{2,\Omega}$	*Order*	$\|u - U - E\|_{2,\Omega}$	*Order*	θ
	40	1.0279	–	$1.4619e-01$	–	1.0249
	320	$5.2283e-01$	1.0155	$4.4524e-02$	1.7651	1.0257
0	1080	$3.4927e-01$	1.0031	$2.1341e-02$	1.8323	1.0215
	2560	$2.6212e-01$	1.0009	$1.2524e-02$	1.8594	1.0181
Without correction						
	40	1.1016	–	$8.8975e-01$	–	0.5326
	320	$5.8671e-01$	0.9290	$4.9958e-01$	0.8104	0.4456
0	1080	$4.0105e-01$	0.9432	$3.4893e-01$	0.8788	0.4097
	2560	$3.0492e-01$	0.9549	$2.6829e-01$	0.9111	0.3901

Computational Examples

- Results of Problem 1 in structured meshes, $p = 1$

$\mathcal{P}_p$ (**Example 1**)						
With correction						
p	N	$\|u-U\|_{2,\Omega}$	*Order*	$\|u-U-E\|_{2,\Omega}$	*Order*	θ
	40	$1.6430e-01$	–	$1.1208e-02$	–	1.0078
1	320	$2.9779e-02$	2.0298	$1.7188e-03$	2.7045	1.0100
	1080	$1.3256e-02$	2.0085	$5.6092e-04$	2.7975	1.0107
	2560	$7.4596e-03$	2.0031	$2.4287e-04$	2.9433	1.0101
Without correction						
	40	$1.4619e-01$	–	$1.2639e-01$	–	0.3583
1	320	$4.4524e-02$	1.7651	$4.0626e-02$	1.6926	0.2940
	1080	$2.1341e-02$	1.8323	$1.9748e-02$	1.7993	0.2835
	2560	$1.2524e-02$	1.8594	$1.1664e-02$	1.8371	0.2794

Computational Examples

- Results of Problem 1 in structured meshes, $p = 3$

$\mathcal{P}_p$ (**Example 1**)						
With correction						
p	N	$\|u - U\|_{2,\Omega}$	$Order$	$\|u - U - E\|_{2,\Omega}$	$Order$	θ
	40	$5.6848e-04$	–	$3.5351e-05$	–	1.0088
3	320	$3.6432e-05$	4.0338	$1.2754e-06$	4.8473	1.0052
	1080	$7.2110e-06$	4.0087	$1.7850e-07$	4.8463	1.0034
	2560	$2.2825e-06$	4.0035	$4.3753e-08$	4.9049	1.0026
Without correction						
	40	$7.1418e-04$	–	$6.2790e-04$	3.7141	0.37205
3	320	$5.3016e-05$	3.7605	$4.8864e-05$	3.6696	0.36086
	1080	$1.1461e-05$	3.7898	$1.0778e-05$	3.7375	0.3509
	2560	$3.7927e-06$	3.8634	$3.6021e-06$	3.8291	0.3505

Computational Examples

Results of Problem 1 in Unstructured meshes, $p = 2, 3$

- Results for solving Example 1 on the second meshes are given in the following table for $p = 0, 1, 2, 3$ using the new method.

Results for solving Example 1 on the second meshes are given in the following table for $p = 0, 1, 2, 3$ using the new method.

$\mathcal{P}_p$ (**Example 1 CM**)					
$p = 2$					
N	$\|u-U\|_{2,\Omega}$	*Order*	$\|u-U-E\|_{2,\Omega}$	*Order*	θ
192	$1.6180e-003$	–	$7.5291e-005$	–	1.0161
934	$4.1263e-004$	3.0445	$1.4901e-005$	3.6954	1.0024
3731	$1.0915e-004$	3.0376	$2.6278e-006$	3.8781	1.0021
8713	$4.6621e-005$	3.2370	$8.7999e-007$	4.3913	1.0014
$p = 3$					
24	$8.3118e-04$	–	$5.2796e-05$	–	1.0229
192	$5.1438e-05$	4.0142	$1.8696e-06$	4.8196	1.0132
934	$9.2356e-06$	4.3114	$2.9574e-07$	4.7873	1.0020
2121	$3.3315e-06$	4.5695	$8.7329e-08$	5.4664	1.0020

Conclusion and future work

- Conclusion:
  - Investigated higher-order DGM for scalar first-order hyperbolic problems on tetrahedral meshes.
  - Construct asymptotically correct a posteriori error estimates for discontinuous finite element solutions
  - Write explicitly the basis functions for the error spaces corresponding to the finite element space $\mathcal{P}_p$.
  - These a posteriori error estimates tested on several linear problems to show their efficiency and accuracy under mesh refinement for smooth solutions.

- Future work

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- Future work
  - Nonlinear problem
  - Transit problem
  - System
  - Other spaces

Thanks!