A Modification to Iterated Tikhonov Regularization for Deconvolutions

Nathaniel Mays

Department of Mathematics
University of Pittsburgh

February 20, 2010

Joint with: M. Zhong

Outline

Differential Filter

Definition (Germano):
Denote the filtering operator on ϕ by $G\phi = \overline{\phi}$ by

$$-\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi \tag{1}$$

- Approximation of the Gaussian filter
- Calculated using local values
- Satisfies $I - G = O(\delta^2)$
- Linear, spacially invariant, isotropic, scale invariant up to δ^2

Tikhonov Regularization

Workhorse: Tikhonov (Lavrentiev) Regularization
Pick $\alpha > 0$

$$(G + \alpha I)\phi_0 = \overline{\phi} + \varepsilon$$

Known:

$$\|\phi_0 - \phi_{true}\| \leq C\alpha + \frac{\varepsilon_0}{\alpha}$$

Iterated Tikhonov Regularization

Iterated Tikhonov (Lavrentiev) Regularization
Pick $\alpha > 0$ and $J \in \mathbf{N}$

$$(G + \alpha I)\phi_0 = \overline{\phi} + \varepsilon$$

For $j = 1, ..., J$

$$(G + \alpha I)(\phi_j - \phi_{j-1}) = \overline{\phi} + \varepsilon - G\phi_{j-1}$$

Known:

$$\|\phi_J - \phi_{true}\| \leq C(J)\alpha^{J+1} + \frac{(J+1)\varepsilon_0}{\alpha}$$

Transfer Functions

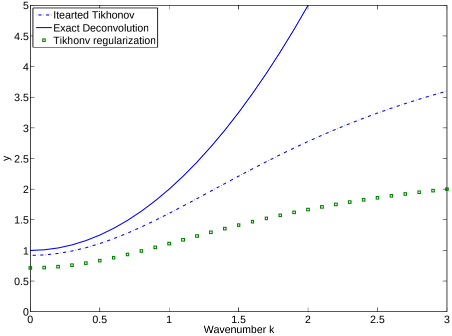


Figure: Comparison of the transfer functions of the exact deconvolution, Tikhonov regularization and iterated Tikhonov regularization all with $\alpha = 0.4$.

Modified iterated Tikhonov regularization

Modified iterated Tikhonov-Lavrentiev Regularization (MITLAR)
Solve for ϕ_0 satisfying

$$((1-\alpha)G + \alpha I)\phi_0 = y \tag{2}$$

For j = 1,...,J, solve for ϕ_j satisfying

$$[(1-\alpha)G + \alpha I](\phi_j - \phi_{j-1}) = y - G\phi_{j-1} \tag{3}$$

Transfer Functions

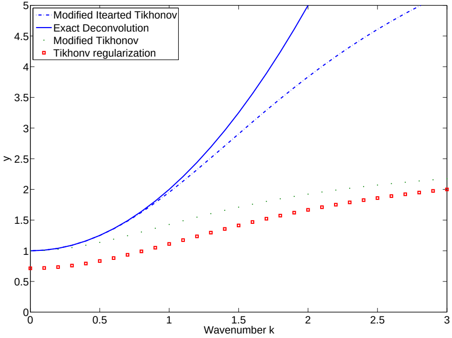


Figure: Comparison of the transfer functions of the exact deconvolution, Tikhonov regularization, modified Tikhonov regularization, and iterated modified Tikhonov regularization, all with $\alpha = 0.4$.

Error - modified regularization

Theorem: [M., Zhong]
Given $\overline{\phi} + \varepsilon$. If $\phi_{true} \in Range(G^J)$ (regularity) for some integer $J > 1$, then ϕ_J given by MITLAR satisfies the following error bound

$$\|\phi_{true} - \phi_J\| \leq C(\alpha^{J+1}\delta^{2J+2} + \frac{\varepsilon_0}{\alpha}) \tag{4}$$

Descent properties of MITLAR

The problem of

$$G\phi_0 = \overline{\phi} + \varepsilon$$

is equivilent to

$$\phi_0 = \arg\min_{\nu \in X} J_\varepsilon(\nu)$$

where $J_\varepsilon(\nu) = \frac{1}{2}(G\nu, \nu)_X - (\overline{\phi} + \varepsilon, \nu)_X$
Proposition: [M., Zhong]

$$J_\varepsilon(\phi_{j+1}) < J_\varepsilon(\phi_j)$$

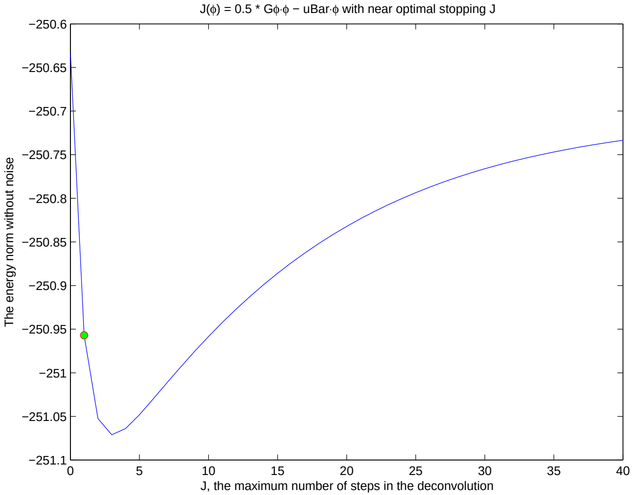
unless $\phi_{j+1} = \phi_j$

Stopping condition

Theorem: [M., Zhong]
The perturbed MITLAR algorithm are a minimizing sequence for the noise free functional as long as

$$\frac{\varepsilon_0}{\|\phi_{j+1} - \phi_j\|} \leq \alpha \leq \frac{1}{2}$$

Stopping condition



Conclusion

Using MITLAR we ...

- have a double asymptotic error bound
- have a bound on noise amplification
- match the transfer function for low wave numbers
- have a stopping condition for iterations