Sensor Location in Feedback Control of Heat Equation

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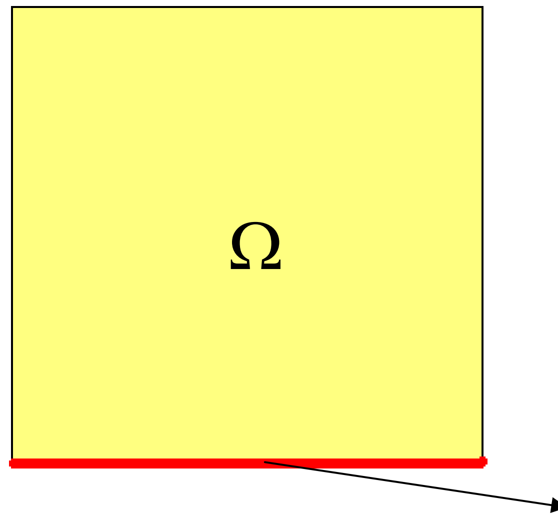
Overview

- Model and problem formulation
    - LQR control
    - Abstract and weak formulation for heat equation
- Approximations and convergence
- Optimal sensor placement
- Numerical results

Model and Problem Formulation



$$\Omega = (0,1)\times(0,1) \subseteq R^2$$

$$\partial\Omega = \Gamma$$

Γ_c : *Controlled boundary*

$$\frac{\partial}{\partial t} w(t,x,y) = c^2 [\frac{\partial^2 w(t,x,y)}{\partial x^2} + \frac{\partial^2 w(t,x,y)}{\partial y^2}] \qquad (1)$$

$$w|_{\Gamma_c} = u(t,x),\ w|_{\Gamma_u} = 0,\ \Gamma_u = \Gamma - \Gamma_c \qquad (2)$$

$$w(0,x,y) = w_0(x,y) \qquad (3)$$

Thermal diffusivity : $c^2 = 0.22167e^{-3} \text{ft}^2 / s.$

LQR $\rightarrow$ feedback operator K

$\xrightarrow{\text{Hilbert-Schmidt}}$ integral representation

$$u(t) = -Kw(t) = -\int_{\Omega} k(\vec{x})w(t,\vec{x})d\vec{x}$$

$\xrightarrow{\text{kernal function } k(\vec{x})}$ functional gain

$\xrightarrow{\text{aprroximated by quadrature (CVT)}}$ sensor locations

$\rightarrow$ reduced system state estimation

LQR Control

Find a control *u* that minimizes

$$J(u) = \int_0^{\infty} [< Qw(t), w(t) >_H + < Ru(t), u(t) >_U]dt \quad (4)$$

subject to

$$\dot{w}(t) = Aw(t) + Bu(t), \quad w(0) = w_0 \in H \quad (5)$$

When the optimal control exists, it is given in feedback form

$$u_{opt}(t) = -Kw_{opt}(t) \qquad (6)$$

K is called *feedback operator*. In particular,

$$K = R^{-1}B^{*}P \qquad (7)$$

where P is the non-negative definite solution to the Algebraic Riccati Equation (ARE)

$$A^{*}X + XA - XBR^{-1}B^{*}X + Q = 0. \qquad (8)$$

Abstract and Weak Formulation for Heat Equation

Define the operator A on the domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \tag{9}$$

by

$$Aw = c^2 \Delta w = c^2 [\frac{\partial^2 w(t,x,y)}{\partial x^2} + \frac{\partial^2 w(t,x,y)}{\partial y^2}] \tag{10}$$

A is Laplacian operator and generates an analytic semigroup $S(t)$ on state space $L^2(\Omega)$, denoted as H. $S(t)$ is exponentially stable, which guarantees the existence of a solution to the LQR problem.

Let W be the space $D(A) = D(A^*)$ with graph norm

$$\|v\|_W = \|A^* v\|_H + \|v\|_H \tag{11}$$

It follows that the injections

$$W \subset H = H' \subset W' \tag{12}$$

are all continuous and dense. Now we lift the operator A to an operator $\widehat{A} : H \to W'$ by

$$[\widehat{A} w] v = < w, A^* v >_H = c^2 \int_\Omega w \cdot \Delta v d\sigma, \tag{13}$$

for all $v \in W$. By semi-group theory ([1], p159, Alain Bensoussan, etc.), we can prove that

$$\widehat{A} : H \to W' \ \ generates\ \widehat{S}(t) : W' \to H \subset W' ,$$

$$\text{and} \quad \widehat{S}(t)|_H = S(t). \tag{14}$$

Let $U = L^2(\Gamma_c)$. Operator $D: U \to L^2(\Omega)$ denotes the Dirichlet map,

$$Du = w. \tag{15}$$

Where w is the solution of the Dirichlet problem

$$\begin{aligned} \Delta w &= 0 \quad \text{in } \Omega \\ w|_{\Gamma_u} &= 0,\ w|_{\Gamma_c} = u. \end{aligned} \tag{16}$$

Then D is bounded from U into $H^{1/2}(\Omega)$.

Further, if one defines $B: U \to W'$ by

$$B = -\widehat{A}D, \qquad (17)$$

B is unbounded operator from control space U to state space H. Then the control problem can be formulated as the well-posed system in W'

$$\dot{w}(t) = \widehat{A}w(t) + Bu(t) \in W' \quad \text{(very weak form)} \qquad (18)$$

$$w(0) = w_0 \in H = D(\widehat{A}).$$

Therefore, system (1)-(3) allows for a unique very weak solution.

$$w(t) = \widehat{S}(t)w_0 + \int_0^t \widehat{S}(t-s)Bu(s)ds \in D(\widehat{A}) = H \qquad (19)$$

Solving ARE (8) yields

$$K = R^{-1}B^{*}P \in L(H,U) \tag{20}$$

Based on the numerical results (see [4,5,6]), we conjecture that there is a kernel $k(\xi, x, y) \in L^{2}(\Gamma_c \times \Omega)$ such that

$$[Kw](\xi) = \int_{\Omega} k(\xi, x, y) w(x, y) dxdy \tag{21}$$

Here the kernel $k(\xi, x, y)$ is called the *functional gain*. Its exact regularity is unknown.

A Variational Form of the Problem

By [2], p.27, we know that

$$< \dot{w}, \varphi > = < (-A)^{1/2} w, (-A)^{1/2} \varphi > + < u, B^* \varphi > \tag{22}$$

is well-defined for any $\varphi \in H_0^1(\Omega)$ and $u \in H^{1/2}(\Gamma_c)$. Therefore, based on (22), we use standard bilinear finite elements with uniform meshes on Ω and “hat” function as defined on Γ_c to approximate our system (18).

Convergence of the Approximating System

Now, system (18) can be replaced by an approximation system:

$$\dot{w}_N(t) = A_N w_N(t) + B_N u(t)$$
$$w_N(0) = w_{0N} \qquad (23)$$

Notes: we select the approximating space complying with the usual approximation properties:

$$\left\|\Pi_N \varphi - \varphi\right\|_{H^l(\Omega)} \le c h^{s-l} \left\|\varphi\right\|_{H^s(\Omega)}, \; s \le 2; s - l \ge 0; 0 \le l \le 1, \; \forall \varphi \in H^s(\Omega), \qquad (24)$$

where $\Pi_N : H \to V_N \subset V$ is the orthogonal projection.

Then, we have the following convergence results (see[1], p.129 and p.133, I. Lasiecka, R.Triggiani):

(1) $$\left\| P_N \Pi_N - P \right\|_{L(H)} \le ch^s, s < 1/2; \qquad (25)$$

(2) $$\left\| K_N - K \right\|_{L(H,U)} \to 0,\ as\ h \to 0; \qquad (26)$$

(3) $$\left\| \widehat{w}_N(t) - w(t) \right\|_H \to 0,\ as\ h \to 0. \qquad (27)$$

Where

$$\widehat{w}_N \mid_{\Omega} = w_N, \widehat{w}_N \mid_{\Gamma_u} = 0, \widehat{w}_N \mid_{\Gamma_c} = u_N.$$

Optimal Sensor Placement

Solving the approximate finite element LQR problem produces a feed back operator K_N with representation

$$\begin{aligned}\left[K_N w\right](\xi) &= \int_{\Omega} k^N(\xi, x, y) w(x, y)\,dxdy \\ &\approx \sum_{i=1}^{M} [k^N(\xi, x_i, y_i) w(x_i, y_i)] \left|\sigma_{\Omega_i}\right|\end{aligned} \qquad (28)$$

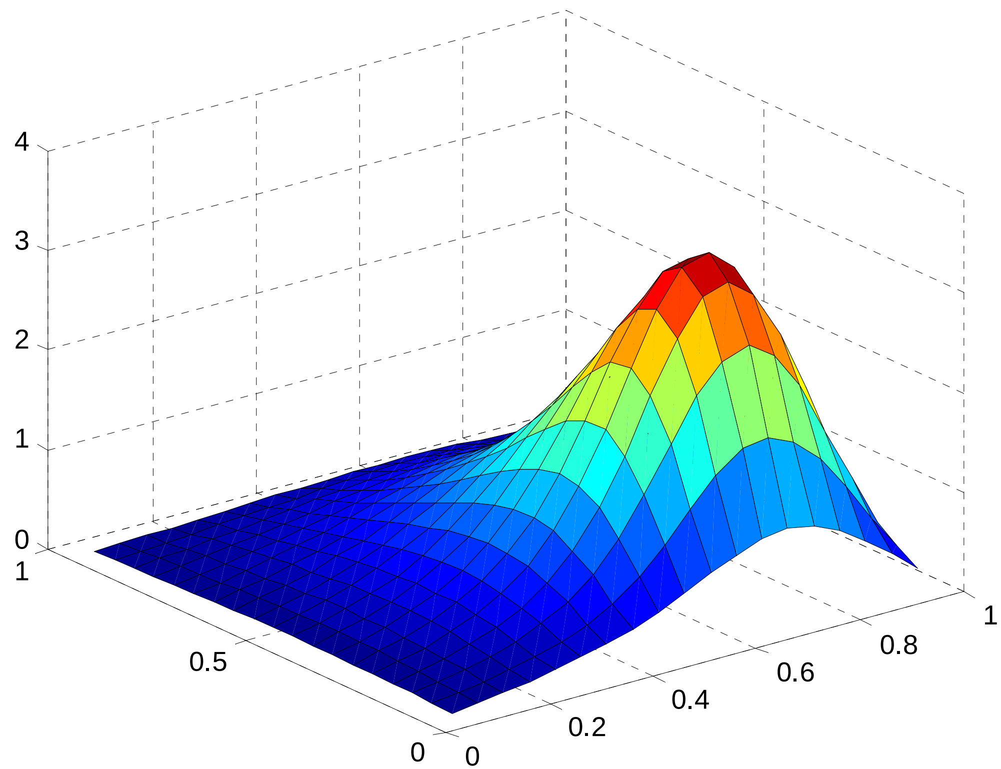
Where functional $k^N(\xi, \cdot, \cdot) \in L^2(\Omega)$ for each N. And (x_i, y_i) is where we put sensor. The discretization yields N-1 discrete actuator locations along Γ_c.

In order to use Centroidal Voronoi tessellation yielding the optimal sensor place (see [9]), we apply the singular value decomposition (SVD) to determine the best approximation, denoted as $k^N(x, y)$, to the set of functional gains.

Let $\vec{z}_i^* = (x_i, y_i)$, then CVT yields

$$\vec{z}_i^* = \frac{\int_{\Omega_i} \vec{z} k^N(\vec{z}) d\vec{z}}{\int_{\Omega_i} k(\vec{z}) d\vec{z}}, i = 1, 2, \ldots, M. \tag{29}$$

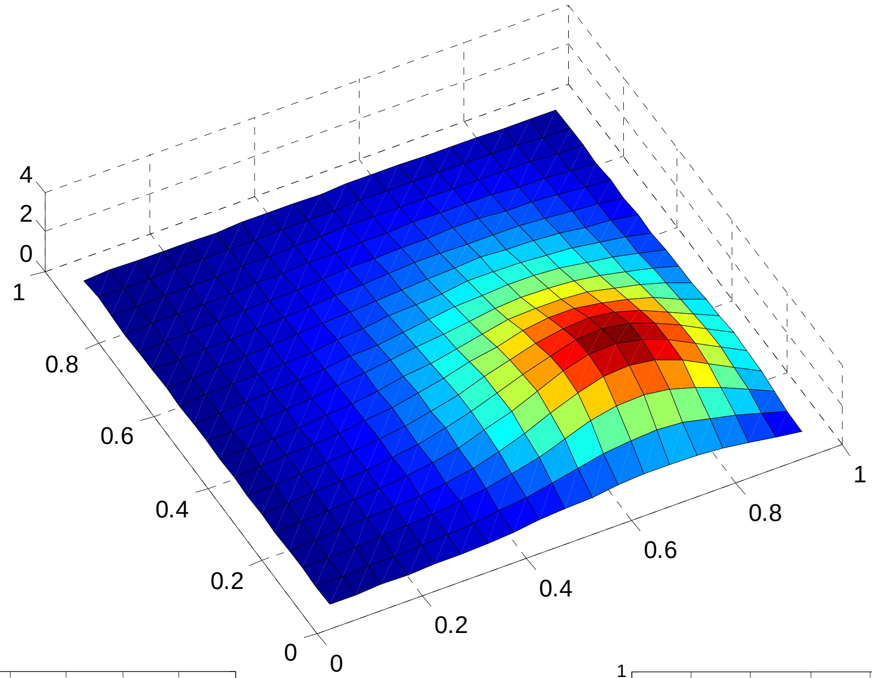
Numerical Results

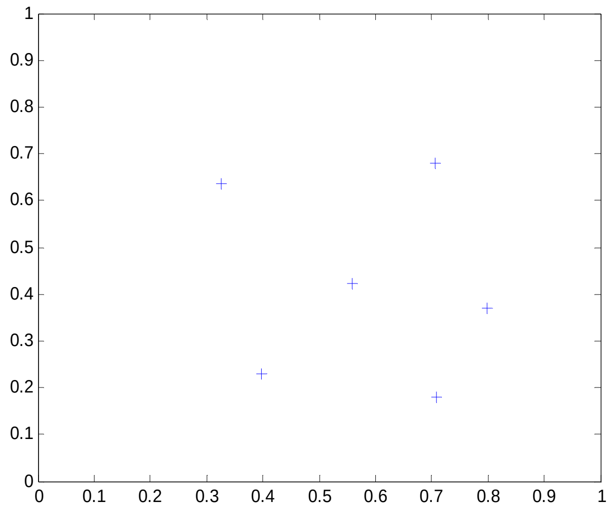


SVD: rank-3 approximations captures 99.23%

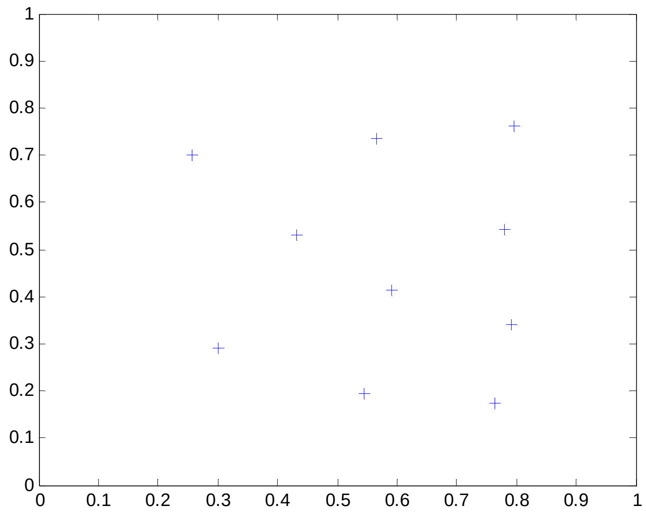
$$N = 20, \quad R = w_r I_U, \; w_r = 1000,$$

$$Q(x,y) = q(x,y)I, \quad q(x,y) = \begin{cases} 2000, & if\, 0.2 \le x \le 0.8, 0.6 \le y \le 0.8 \\ 10, & otherwise \end{cases}$$

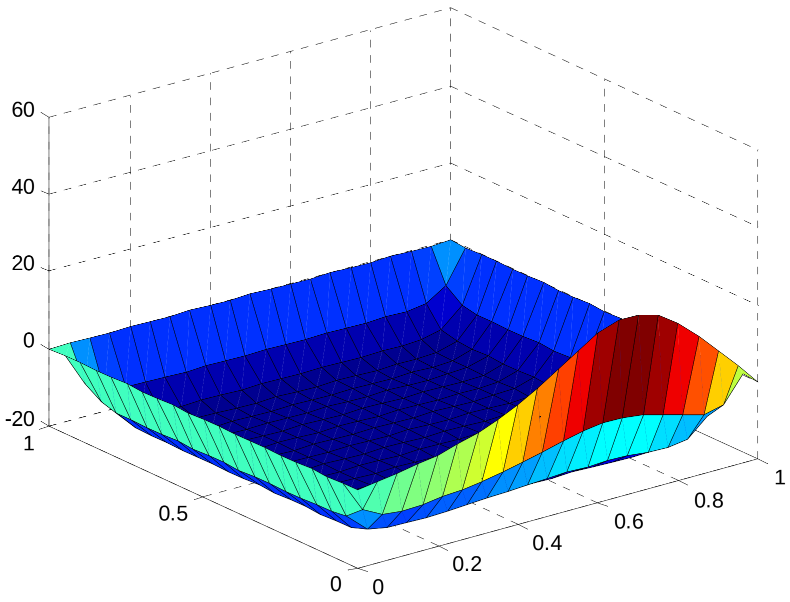




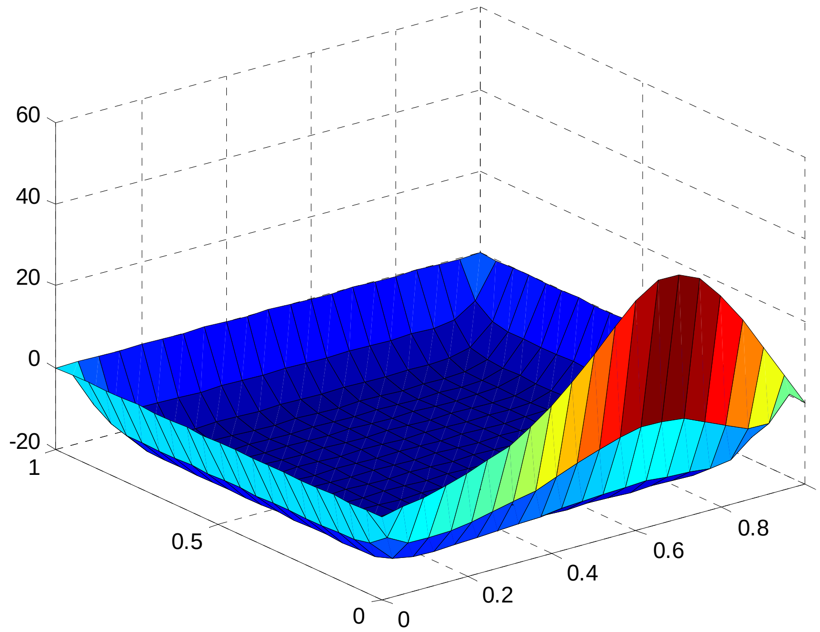
M_sensors=6



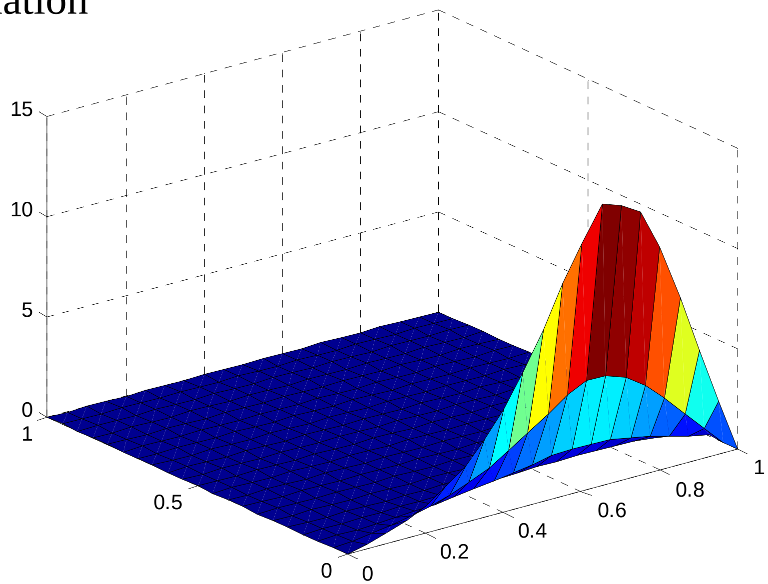
M_sensors=10



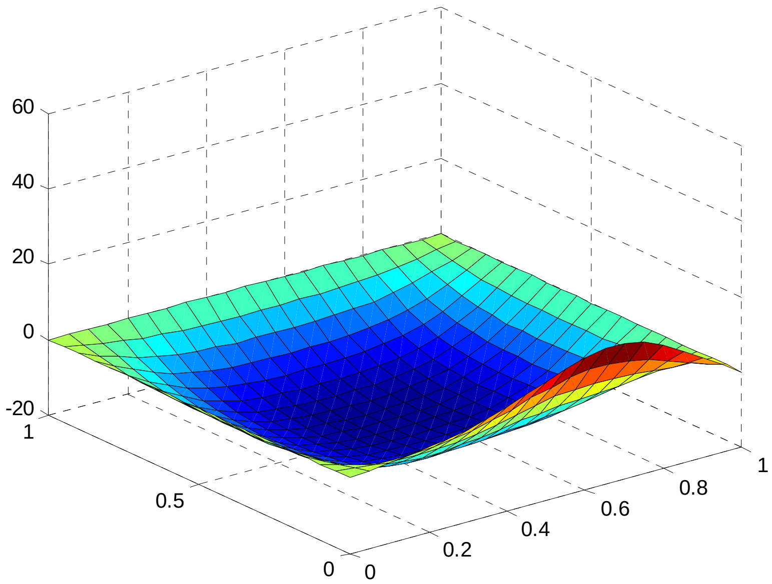
$w_0 = -20$, T=10s
Close-Loop full order simulation



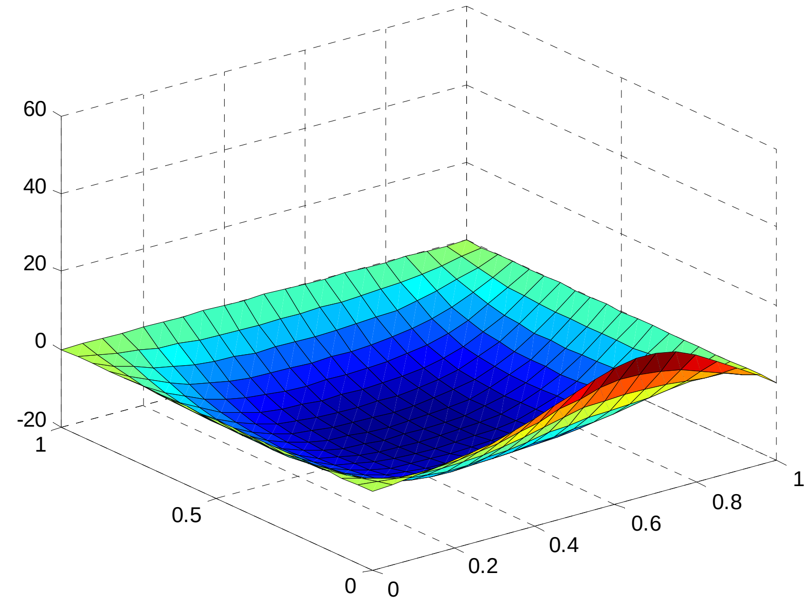
M_sensors=6. Closed-Loop reduced order simulation



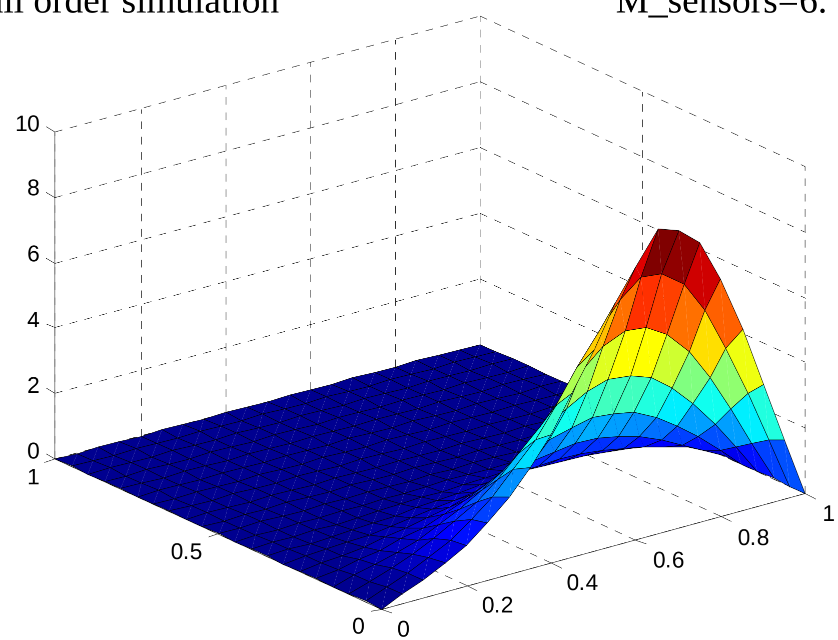
Comparison: reduced order – full order system



T=100s. Closed-loop full order simulation

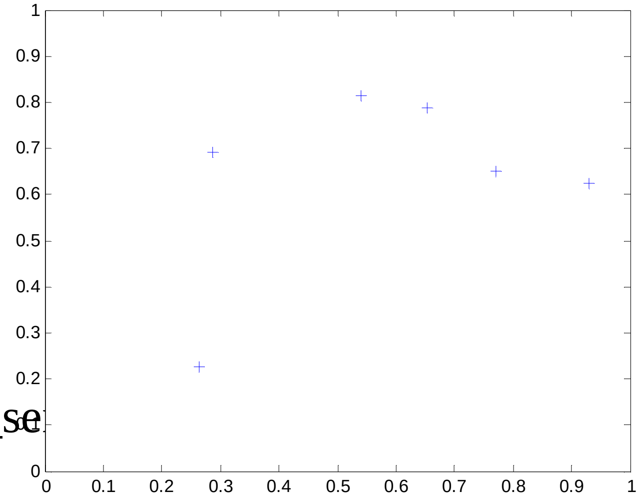


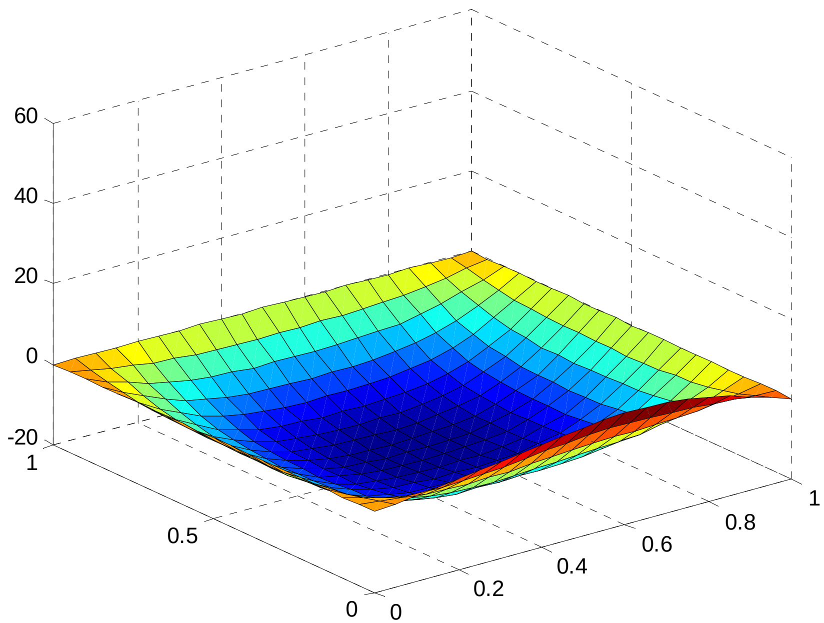
M_sensors=6. Closed-loop reduced order simulation

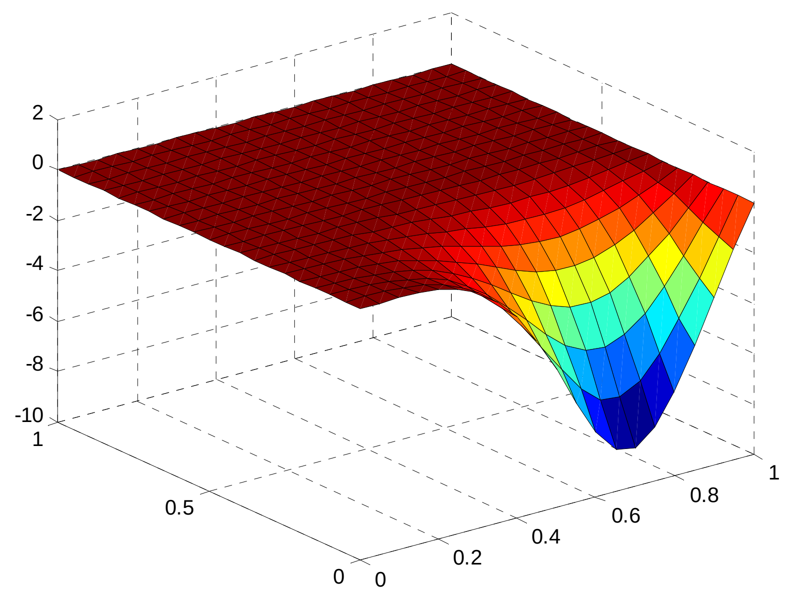


Comparison: reduced order – full order system

M_se







M_sensors=6.

Closed-loop reduced system simulation

Comparison: reduced order- full order system

Future Work

1. Is feedback operator *K* Hilbert- Schmidt, when *B* is unbounded ? How about the regularity of the kernel function $k(\xi, x, y)$?

2. Any other methods to approximate the representation for feedback operator and hence give the optimal sensor locations?

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Thank you!