Finite Element approximations to the Navier-Stokes equations with Scott-Vogelius elements

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Overview

- Joint Work with:
  - Vince Ervin & Leo Rebholz (Clemson University).
  - Alexander Linke
    (Weierstrass Institute for Applied Analysis & Stochastics)
- Supported by NSF grant DMS0914478
- Mass conservation and the Scott-Vogelius(SV) element
  - Structure of SV element.
  - Time dependent NSE results.
  - Comparison with Grad-Div Stabilization.
  - Conclusions and Future Work.

Notation

$(\cdot,\cdot)$ $L^2(\Omega)$ inner product, $\|\cdot\|$ $L^2(\Omega)$ norm.

$\mathbf{X}_h = \{\mathbf{v}_h \in [C(\Omega)]^d : \mathbf{v}_h|_T \in \mathbf{P}_k(T) \ \ \forall T \in \mathcal{T}_h\}.$

$Q_h^{TH} = \{q_h \in L_0^2 \cap C(\Omega) : q_h|_T \in P_{k-1}(T) \ \ \forall T \in \mathcal{T}_h\}.$

$Q_h^{SV} = \{q_h \in L_0^2 : q_h|_T \in P_{k-1}^{-1}(T) \ \ \forall T \in \mathcal{T}_h\}.$

$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h : (\nabla \cdot \mathbf{v}_h, q_h) = 0 \ \ \forall q_h \in Q_h\}.$

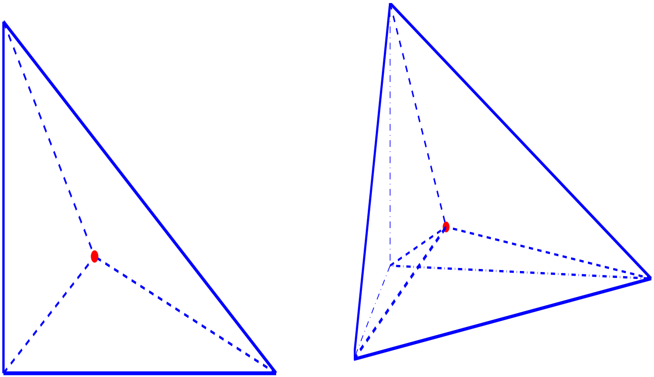
$(\mathbf{X}_h, Q_h)$ is LBB stable.

Ω is convex polygon/polyhedra.

Conservation of Mass

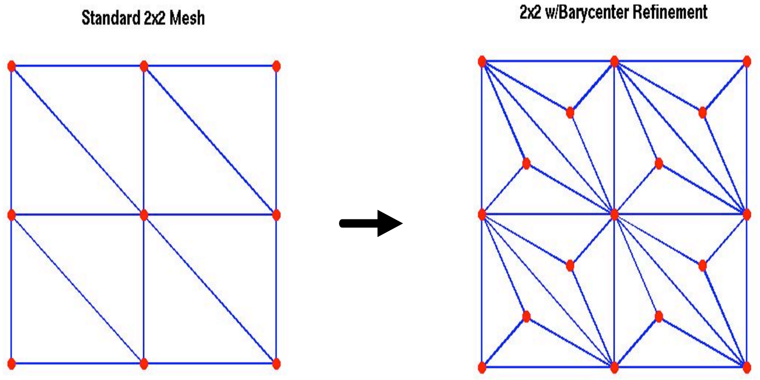
- Stokes, NSE, MHD and other physical equations require, $\nabla \cdot \mathbf{u} = 0$ (incompressibility).
- Weak enforcement of $\nabla \cdot \mathbf{u} = 0$ via $(\nabla \cdot \mathbf{u}_h, q_h) = 0 \;\; \forall q_h \in Q_h$, can result in $\|\nabla \cdot \mathbf{u}_h\| >> 0$ and poor mass conservation for common element pairs.
- SV pair, $(\mathbf{P}_k, P_{k-1}^{\text{disc}}) \Rightarrow \|\nabla \cdot \mathbf{u}_h\| = 0$,
  - LBB stable for $k \geq d$ on barycenter refined quasi-unif. mesh (L.R. Scott & M. Vogelius, S. Zhang 2004).
  - LBB essential for stability & uniqueness.
  - Extensions to:
    - Stokes (S. Zhang 2004)
    - Steady NSE & Oseen equation (Linke 2007)

Barycenter Refinement in 2d & 3d



- In a 2d refinement(left) we triple the number of triangles. With 3d(right) we quadruple the number of tetrahedra.

2D barycenter refinement



Related work on Exact Mass-Conservation

- $\mathbf{Q}_{k+1,k} \times \mathbf{Q}_{k,k+1}$ elements (Zhang 2009).
- $(\mathbf{P}_1, \text{div } \mathbf{P}_1)$ w/2D Powell-Sabin Structure (Zhang 2009).
- Cockburn, Kanschat, & Schötzau (2007) looking at Div-free approach with DG methods applied to the NSE.
  - Requires $\mathbf{u} \in \mathbf{H}(\text{div})$.

What are we giving up?

- $\dim(Q_h^{SV}) > \dim(Q_h^{TH})$, hence a more costly linear solve.
  - Extent of additional cost is being studied.
  - SV elements have competitive matrix assembly.
- $\dim(\mathbf{V}_h) = \dim(\mathbf{X}_h) - \dim(Q_h)$
- SV elements are working with a smaller $\mathbf{V}_h$ space when compared to Taylor-Hood(TH) on the same mesh. Thus for better physics wrt incompressibility we are giving up dof.

Stokes Results

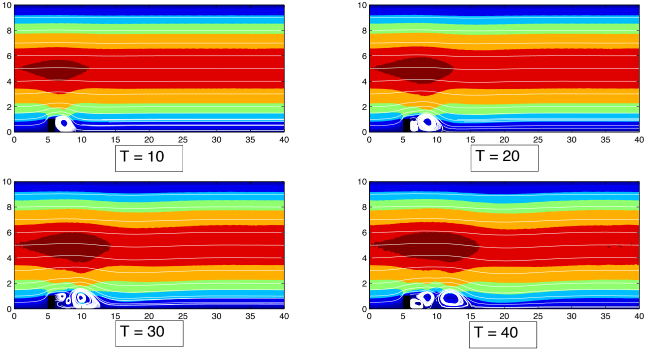
$$\mathbf{u} = [\cos(N\pi z), \sin(N\pi z), \sin(N\pi x)], p = \cos(N\pi(x+y)).$$

Table: Sample Problem Velocity Error, $\nu = 1.0$.

Elem.	N	DOF	L^2 Error	H^1 Error	$\|\text{div } \mathbf{u}_h\|$
TH32	2	39486	0.0058	0.1914	0.1549
SV32	2	39231	3.89e-4	0.0071	5.151e-14
TH32	8	39486	0.0678	1.4555	0.4165
SV32	8	39231	0.1021	2.029	4.795e-14

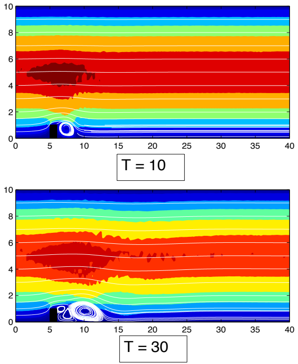
- SV32 elements done on 4x4x4 w/barycenter refinement. TH32 elements done on 6x4x4 w/barycenter refinement.

Time dependent 2D Flow over step (SV21)



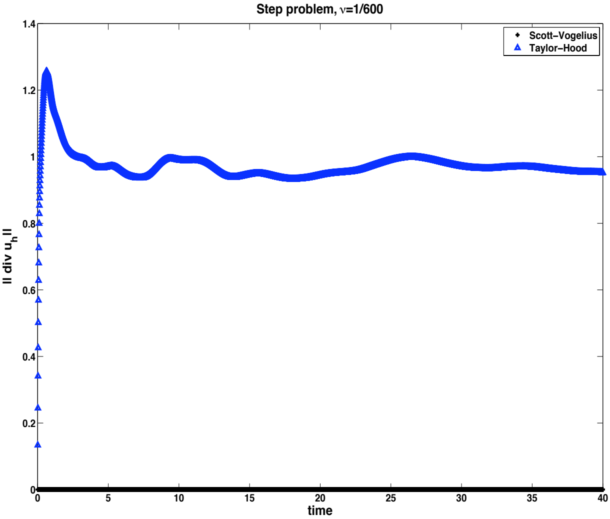
- (7,414 vel. dof, 5,418 pres. dof, $Re = 600$)
- Correct velocity profile
  (Layton, Manica, Neda & Rebholz 2007)

Time dependent 2D Flow over step (TH21)



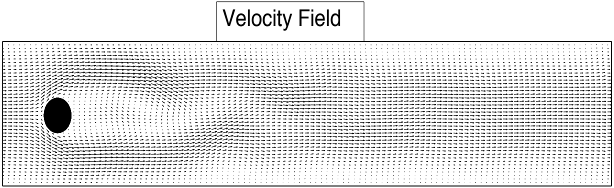
- Less than correct velocity profile.
- (7,414 vel. dof, 951 pres. dof, $Re = 600$)

Time dependent 2D Flow over step

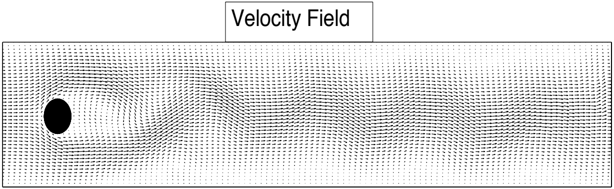


Time dependent 2D Flow around Cylinder

- Incorrect(TH21, 6,578 vel. dof, 845 pres. dof)

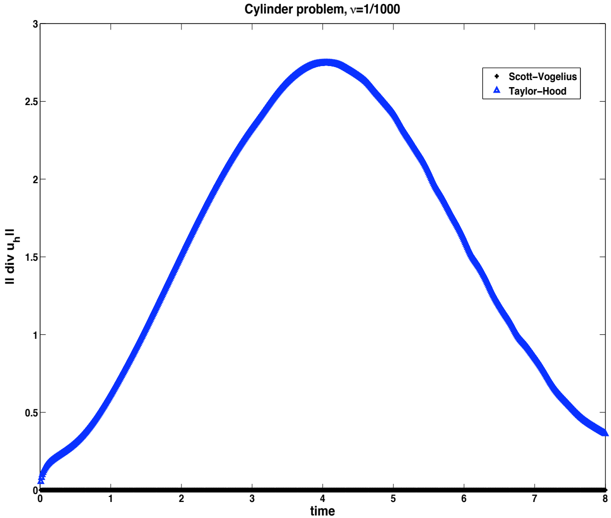


- Correct(SV21, 6,578 vel. dof, 4,797 pres. dof)



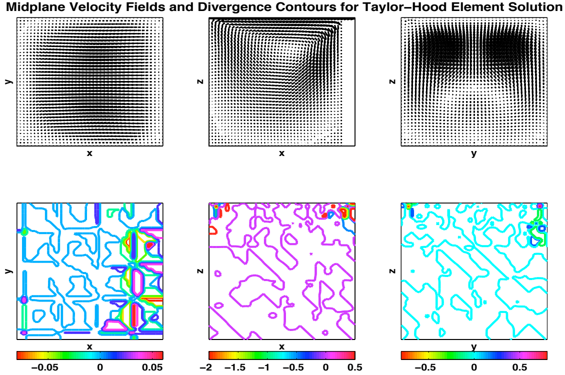
- $0 \leq \mathrm{Re}(t) \leq 100, t = 7.$

Time dependent 2D Flow around Cylinder



3D Lid-Driven Cavity(TH32)

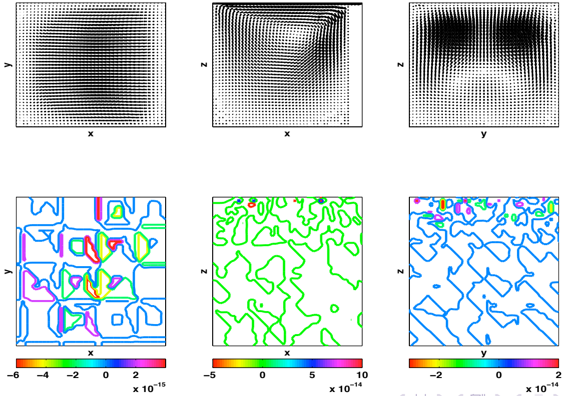
- Re = 100, 57,804 vel. dof, 6,360 pres. dof
- Velocity profiles appear correct (Wong & Baker 2002).
- div **u** appears to be suspect



3D Lid-Driven Cavity(SV32)

- Re = 100, 57,804 vel. dof, 37,840 pres. dof.
- Immense improvement in div **u**.

Midplane Velocity Fields and Divergence Contours for Scott–Vogelius Element Solution



Grad-Div Stabilization

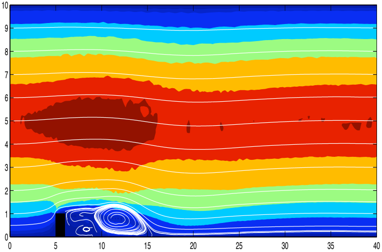
- Grad-Div Stabilization
  - Derived from adding $-\gamma\nabla(\overbrace{\nabla \cdot \mathbf{u}}^{0})$ to momentum equation.
  - (Olshanksii 2000, 2003)
    (Layton, Manica, Neda, Olshanskii & Rebholz, 2009)
  - Can improve conditioning of linear system.
    (Olshanskii 2004)
- Taylor-Hood does a poor job of conserving mass.
- Scott-Vogelius conserves mass exactly.
  - Extra cost.
- Taylor-Hood w/grad-div can improve mass conservation.

SV elements relation to Grad-Div Stab?

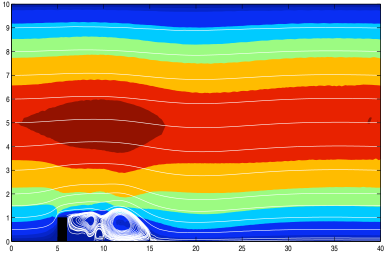
- In general, as γ increases the quality of $\mathbf{u}_h$ can diminish. (Olshanskii 2004)
- Conjecture: On a barycenter refined mesh, larger γ ($\gamma > O(1)$) does a respectable job of conserving mass.

$$\mathbf{V}^{TH+\gamma} \approx \mathbf{V}^{SV}?$$

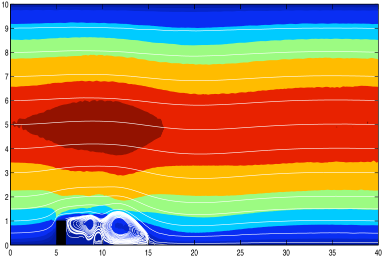
Taylor–Hood, γ=0, T=40



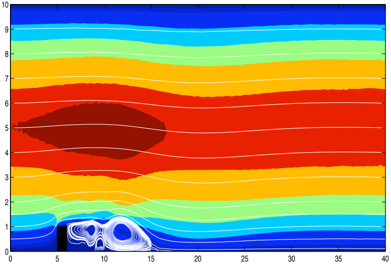
Taylor–Hood, γ=1, T=40



Taylor–Hood, γ=100, T=40

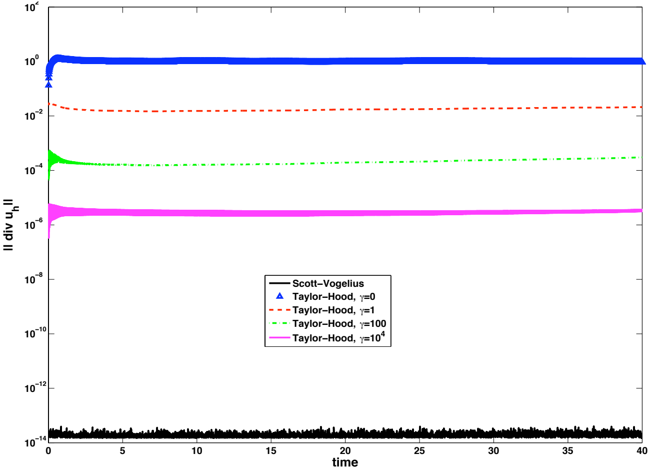


Taylor–Hood, γ=10,000, T=40



Mass Conservation Comparison

- Similar pattern for cylinder & lid-driven cavity problem.



Conclusions & Future Work

- Mass conservation is not good for $(\mathbf{P}_k, P_{k-1})$ elements.
  - Can $\mathbf{u}_h$ have physical meaning?
- Scott Vogelius elements improve mass conservation.
  - What is the extent of this cost?
    - Preconditioner development.
  - Is grad-div stabilization good enough?
    - $\mathbf{f} = 0$ and $\nabla \cdot \mathbf{u}_h^0 = 0$ pointwise.
  - For larger scale problems does anything change (i.e. channel flow, 3D step)
- Extend Scott-Vogelius elements to physical problems which depend heavily on incompressibility condition (i.e. MHD).
- Apply Scott-Vogelius elements to other FEM numerical schemes (i.e Energy/Helicity dual-conserving scheme).

Questions