Quadratic Immersed Finite Elements for interface problems, using Discontinuous Galerkin method

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Outline

1. The model problem
2. Motivation for IFE
3. The Quadratic IFE basis functions
4. The DG formulation of the problem using NIPG method
5. Numerical results

Outline

The model problem

Model interface problem:

$$\begin{cases} -\nabla(\beta\nabla u) = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u_{|\partial\Omega} = g \end{cases}$$

Characteristic of the domain:

$$\beta(x,y) = \begin{cases} \beta^+ & \text{on } \Omega^+ \\ \beta^- & \text{on } \Omega^- \end{cases}$$

Physical jump conditions:

$$[u]_\Gamma = 0\ ; \quad [\beta\frac{\partial u}{\partial n}]_\Gamma = 0$$

$[v]_\Gamma = v^+ - v^-$: jump of v across Γ.

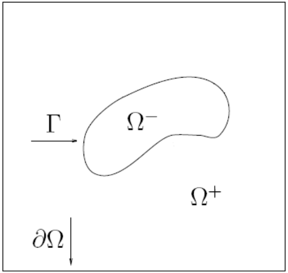
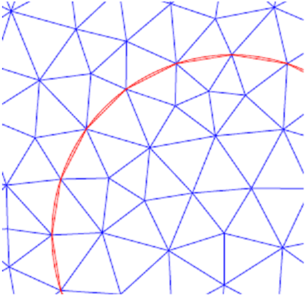


Figure: The domain Ω

Motivation

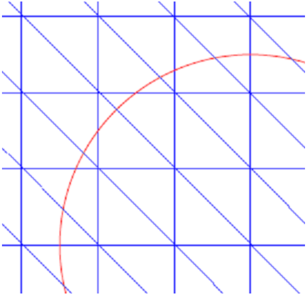
Conventional Finite Elements:

- Basis functions independent from the problem;
- The mesh formed according to the problem (body-fit mesh).



Immersed Finite Elements:

- basis functions constructed according to the problem;
- mesh independent from the problem



Motivation

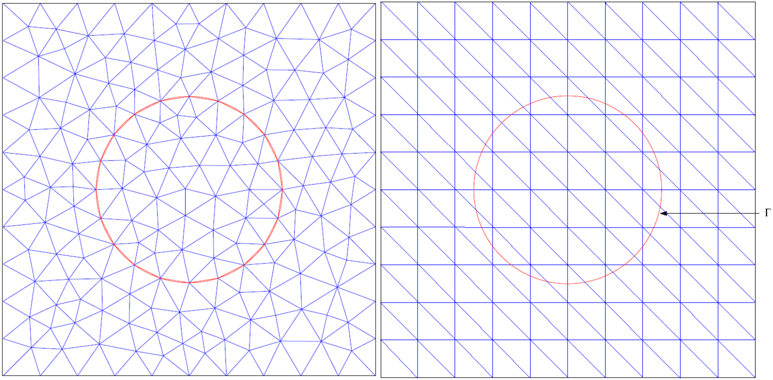


Figure: A "body-fit mesh" and a "mesh with immersed interface" in the case of moving interface

Motivation

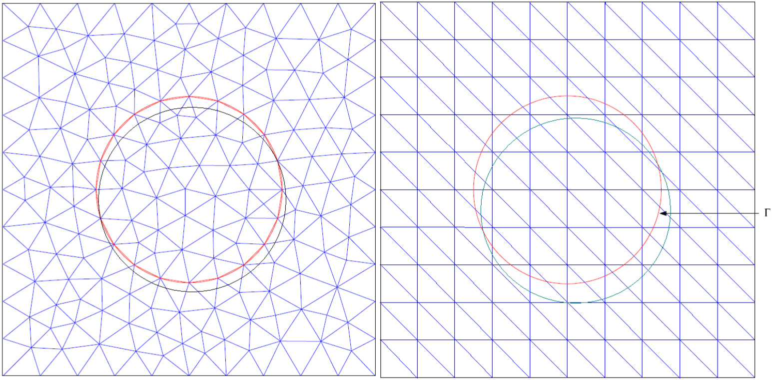
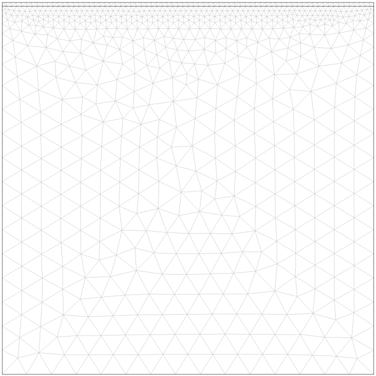


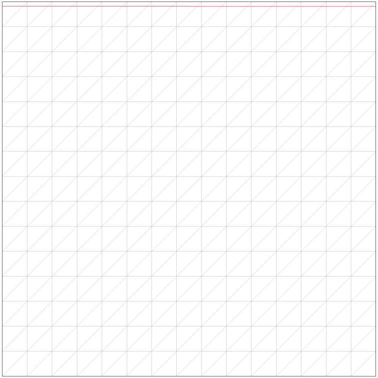
Figure: A "body-fit mesh" and a "mesh with immersed interface" in the case of moving interface

Motivation

A "body-fit mesh" and a "mesh with immersed interface"
(layer of thickness $\frac{1}{100}$ & maximum step size $h_{max} = 1/16$)



DOF=4111



DOF= 1089

The Quadratic IFE basis functions

Reference element: Triangle $\hat{\mathrm{T}} = (0,0),(1,0),(0,1)$.

- $L_i(\hat{x},\hat{y})$: standard Quadratic basis functions on the reference triangle $\hat{T}$, are known. $(i = 1..6)$
- $L_i(V_j) = \delta_{ij}; \quad i,j = 1..6.$
- **Ex:** $L_1(\hat{x},\hat{y}) = 2(1-x-y)(\frac{1}{2}-x-y)$

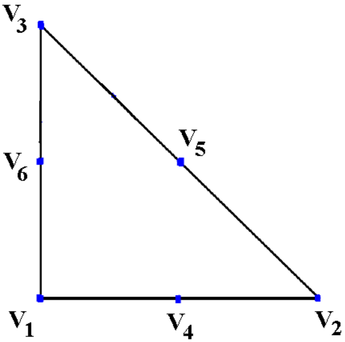


Figure: The Standard Reference triangle

Reference element: Triangle $\hat{\mathrm{T}} = (0,0),(1,0),(0,1)$.

$$\hat{\varphi}_i(\hat{x}) = \begin{cases} \hat{\varphi}_i^1(\hat{x}) = \sum_{j=1}^{6} c_{i,j}^1 L_i(\hat{x},\hat{y}) \\ \hat{\varphi}_i^2(\hat{x}) = \sum_{j=1}^{6} c_{i,j}^2 L_i(\hat{x},\hat{y}) \end{cases}$$

$L_i(\hat{x},\hat{y})$: standard Quadratic basis functions on the reference triangle $\hat{T}$.

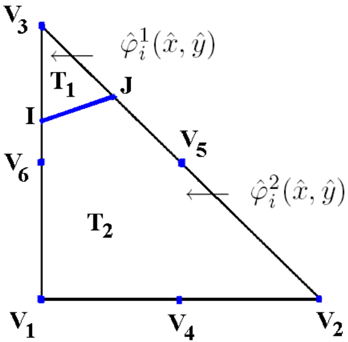


Figure: Immersed Reference triangle

The Quadratic IFE basis functions

Conditions:

- The nodal values

$$\hat{\varphi}_i(V_j) = \delta_{ij}, \quad j = 1..6 \implies 6 \text{ equations}$$

- The physical jump conditions:

$$[\hat{\varphi}_i]_\Gamma = 0 \implies 3 \text{ equations}$$

$$[\beta \frac{\partial \hat{\varphi}_i}{\partial n}]_\Gamma = 0 \implies 2 \text{ equations}$$

Extra-condition:

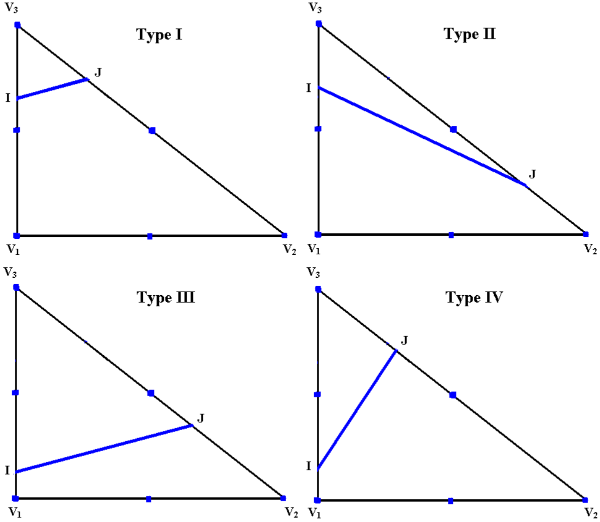
- 1st possibility: (natural assumption)

$$[\beta \Delta \hat{\varphi}_i]_\Gamma = 0$$

- 2nd possibility: (weaker condition)

$$[\beta \frac{\partial^2 \hat{\varphi}_i}{\partial n^2}]_\Gamma = 0$$

4 Types of reference element:



Existence & uniqueness of the shape functions on elements of type I

The interface jump conditions:

$$[\hat{\varphi}_i]_{\hat{\Gamma}} = 0$$

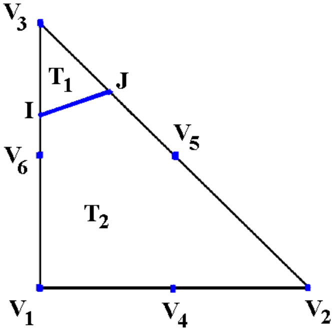
$$[\beta \frac{\partial \hat{\varphi}_i}{\partial n}]_{\hat{\Gamma}} = 0$$

$$[\beta \frac{\partial^2 \hat{\varphi}_i}{\partial n^2}]_{\hat{\Gamma}} = 0$$

Define the sets of indices:

$\mathscr{I}_1 = \{3\}$,

$\mathscr{I}_2 = \{1, 2, 4, 5, 6\}$.



Existence & uniqueness of the shape functions on elements of type I

For $i \in \mathscr{I}_1$ ($i.e.$ $i = 3$), the shape function $\hat{\varphi}_3$ is written as:

$$\hat{\varphi}_3(\hat{x},\hat{y}) = \begin{cases} \hat{\varphi}_3^2(\hat{x},\hat{y}) = c_3 L_3(\hat{x},\hat{y}) & , \text{ on: } \hat{T}^2 \\ \hat{\varphi}_3^1(\hat{x},\hat{y}) = r\ \hat{\varphi}_3^2(\hat{x},\hat{y}) + (1-r)\ \hat{\varphi}_3^2(p(\hat{x},\hat{y})) & , \text{ on: } \hat{T}^1 \end{cases}$$

$p(\hat{x},\hat{y})$ orthogonal projection of $(\hat{x},\hat{y})$ on $\hat{\Gamma}$; $r = \frac{\beta^2}{\beta^1}$.

$\hat{\varphi}_3(V_3) = 1 \Longrightarrow$ one equation for c_3 $(\alpha . c_3 = 1, \alpha \neq 0)$
$\Longrightarrow$ c_3 exists & unique.

Existence & uniqueness of the shape functions on elements of type I

For $i \in \mathscr{I}_2 = \{1, 2, 4, 5, 6\}$, the shape function $\hat{\varphi}_i$ is written as:

$$\hat{\varphi}_i(\hat{x}, \hat{y}) = \begin{cases} \hat{\varphi}_i^1(\hat{x}, \hat{y}) = \sum_{j \in \mathscr{I}_2} c_j L_j(\hat{x}, \hat{y}) & , \text{ on: } \hat{T}^1 \\ \hat{\varphi}_i^2(\hat{x}, \hat{y}) = r\ \hat{\varphi}_i^1(\hat{x}, \hat{y}) + (1 - r)\ \hat{\varphi}_i^1(p(\hat{x}, \hat{y})) & , \text{ on: } \hat{T}^2 \end{cases}$$

$p(\hat{x}, \hat{y})$ orthogonal projection of $(\hat{x}, \hat{y})$ on $\hat{\Gamma}$; $r = \frac{\beta^1}{\beta^2}$.

$\hat{\varphi}_i(V_j) = \delta_{ij} \implies \mathbf{M}\ \mathbf{c} \ = \ \mathbf{e_i}$.

where: $\mathbf{c} = (c_1, c_2, c_4, c_5, c_6)^t$

$$\mathbf{M} = r\ \mathbf{I_d} + (1 - r)\ \tilde{\mathbf{M}}$$

Existence & uniqueness of the shape functions on elements of type I

The eigenvalues of $\mathbf{M}$ are

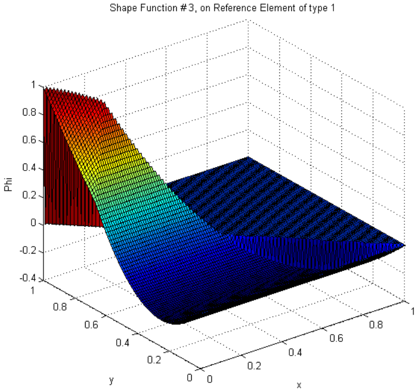
$$\lambda_k = r + (1-r)\,\tilde{\lambda}_k\ , \quad k = 1..5$$

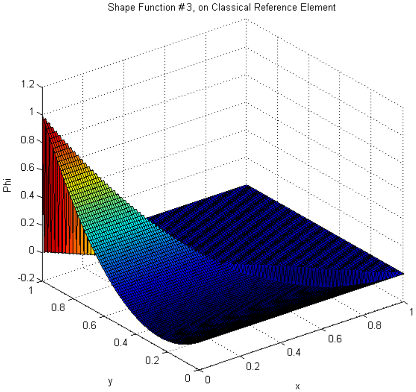
with $r > 0$; $\tilde{\lambda}_k \in [0,1]$, $\forall k = 1..5$,

$\Longrightarrow\ \forall k = 1..5\ :\quad \lambda_k \neq 0$

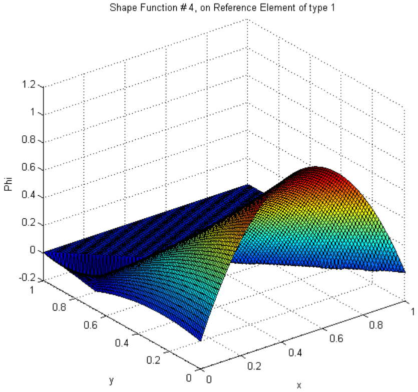
$\Longrightarrow\ \mathbf{M}$ is invertible.

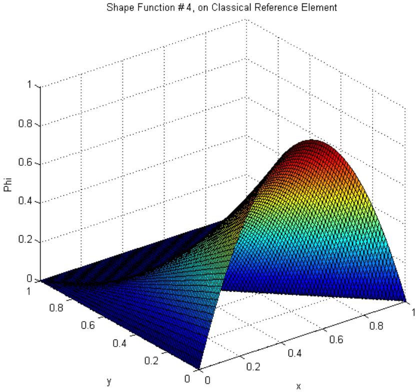
Comparison of FE basis functions and the IFE basis functions of type I



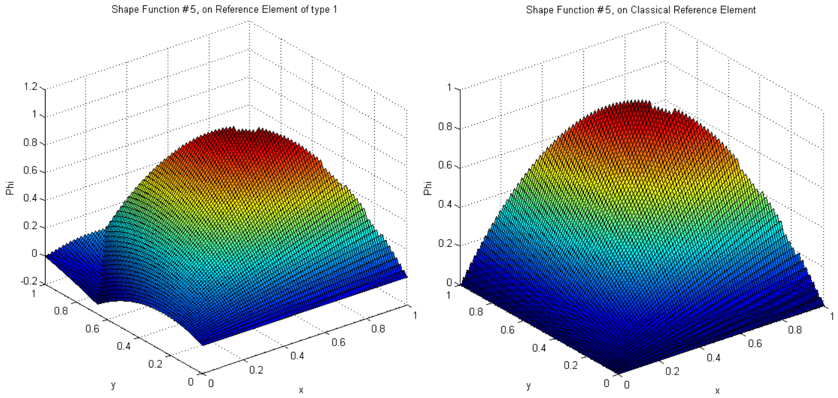


Comparison of FE basis functions and the IFE basis functions of type I





Comparison of FE basis functions and the IFE basis functions of type I



The DG formulation of the problem

The model interface problem:

$$\begin{cases} -\nabla(\beta\nabla u) = f & \text{in } \Omega \\ u_{|\partial\Omega} = g \end{cases} \tag{1}$$

The solution is in: $H^2_E(\Omega) = \{u \in H^2(\Omega) \ / \ u_{|\partial\Omega} = g\}$.

Let: $\mathscr{T}_h$ a triangulation of Ω, with mesh step size h.

$\mathscr{E}^i_h$ set of the interfacial edges on $\mathscr{T}_h$.

NIPG formulation

Find $u \in H^1_E(\mathscr{T}_h)$ such that:

$$(\beta\nabla u, \nabla v)_{\mathscr{T}_h} - < \{\beta\nabla u\}, [v] >_{\mathscr{E}_h^i} + < [u], \{\beta\nabla v\} >_{\mathscr{E}_h^i} + s_1 < [u], [v] >_{\mathscr{E}_h^i} = (f, v)_{\mathscr{T}_h}, \quad \forall v \in H^1_0(\mathscr{T}_h) \qquad (2)$$

where: $H^1(\mathscr{T}_h) = \{v \in L^2(\Omega) \ / \ v_{|T} \in H^1(T) \ \ \forall T \in \mathscr{T}_h\}$

$$< u, v >_{\mathscr{T}_h} = \sum_{T \in \mathscr{T}_h} \int_T u \, v \; dxdy \quad ;$$

$$< u, v >_{\mathscr{E}_h^i} = \sum_{e \in \mathscr{E}_h^i} \int_e u \, v \; dxdy$$

Numerical results

Test function $u(x,y) =$

$$\begin{cases} \dfrac{(6x^2+6xy-4x+3)\ \cos((y-\frac{2}{3})^2-x^2)+(2+3x-3y)\ \sin(\frac{2}{3}-x-y)}{3\beta^+}\ ; & \text{on } \Omega^+ \\ \\ \dfrac{(\frac{\beta^-}{\beta^+}-1)(3-8x+12xy)+(6x^2+6xy-4x+3)\cos((y-\frac{2}{3})^2-x^2)+(2+3x-3y)\sin(\frac{2}{3}-x-y)}{3\beta^-}\ ; & \end{cases}$$

Interface $\Gamma : y = x + \frac{2}{3}$

Domain Square of vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$

Jumps Computation for $r = \frac{\beta^+}{\beta^-} = 5$; then for $r = \frac{\beta^+}{\beta^-} = 1000$.

Using a continuous formulation / Laplacian continuity / r=5

h	$\|\|u - U^h\|\|_2$	RoC
1/4	1.895109e-03	N/A
1/8	4.842859e-04	1.968349
1/16	6.071277e-05	2.995787
1/32	5.041338e-05	0.268193
1/64	6.476201e-06	2.960587

Table: Norm of the FE approximation error and the Rate of Convergence with a jump $r = 5$.

Numerical results

Using DG formulation / Second normal derivative continuous / r=5

h	$\|\|u - U^h\|\|_2$	RoC
1/4	1.850653e-03	N/A
1/8	2.355201e-04	2.974112
1/16	3.014075e-05	2.966063
1/32	3.899650e-06	2.950298
1/64	5.462242e-07	2.835779

Table: Norm of the FE approximation error and the Rate of Convergence with a jump $r = 5$.

Using DG formulation / Laplacian continuity / r=5

h	$\|\|u - U^h\|\|_2$	RoC
1/4	2.185942e-03	N/A
1/8	2.746045e-04	2.992828
1/16	3.426103e-05	3.002714
1/32	4.284828e-06	2.999259
1/64	5.355157e-07	3.000236

Table: Norm of the FE approximation error and the Rate of Convergence with a jump $r = 5$.

Using DG formulation / Laplacian continuity / r=1000

h	$\|\|u - U^h\|\|_2$	RoC
1/4	2.008170e-03	N/A
1/8	2.340102e-04	3.101237
1/16	2.891611e-05	3.016626
1/32	3.584639e-06	3.011973
1/64	4.473713e-07	3.002283

Table: Norm of the FE approximation error and the Rate of Convergence with a jump $r = 1000$.

Consider a thin layer of thickness $e << 1$, bounded to a rectangular domain of thickness 1.

- **Numerical example**:

Jump $= \frac{\beta^+}{\beta^-} = 5$,

thickness $e = 10^{-3}$.

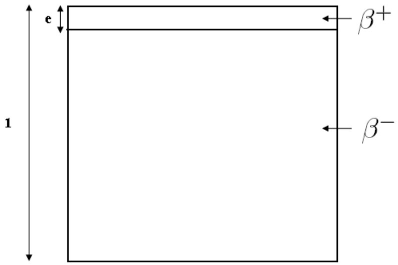
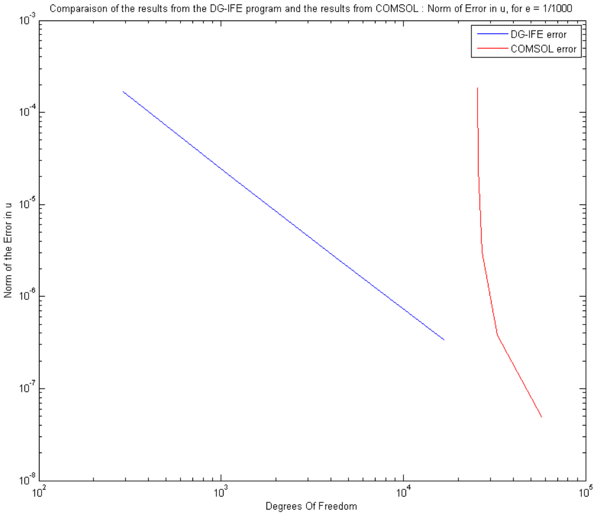


Figure: Geometry of the domain used to compare DG-IFE program & COMSOL results.

Application to thin layer: Comparison with COMSOL



	DOF
IFE	1089
CFE	25706

Table: The minimum DOF required to guarantee an error $< 2.5 \times 10^{-5}$.

Conclusions & future work

- Numerical results show optimal convergence (order $= p + 1 = 3$)
- DG method is necessary to ensure the optimal
- Convergence guaranteed (numerically) for any jump convergence
- Futur work : theoretical proof for the convergence of the method; approximation of the interface by a quadratic curve.

Thank you for your attention