Optimal control of 'harvesting after growth' in an integrodifference population model

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Outline

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1. Integrodifference Equations
2. Optimal control on harvesting problem modeled by integrodifference equations

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3. Comparison of 2 ways to do harvesting (Numerical result)

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3. Comparison of 2 ways to do harvesting (Numerical result)
4. Future Work

Goal

Investigate optimal control theory for integrodifference equations

Integrodifference Equations

Integrodifference

- Discrete in time
- Continuous in space

General equation

$$N_{t+1}(x) = \int_{\Omega} k(x,y) f(N_t(y), y) dy$$

Compare to reaction-diffusion equations

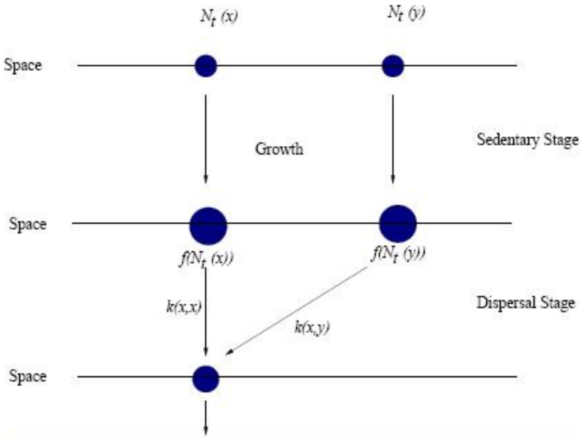
$$N_t - D\frac{\partial^2 N}{\partial x^2} = f(N)$$

- Continuous in both space and time

Why Use Integrodifference Equations

- Can be used to model populations with discrete non-overlapping generations and separate growth and dispersal stages
- Can include a variety of dispersal mechanisms
- Can do a better job of estimating the speed of invasion than reaction-diffusion equations (Mark Kot 2003)

Generating the Integrodifference Model



$$\boxed{N_{t+1}(x) = \int_\Omega k(x,y) f(N_t(y))\, dy}$$

Dispersal Kernels

- Laplace Kernel

$$k(x,y) = \frac{1}{2}\alpha \exp(-\alpha\,|x-y|)$$

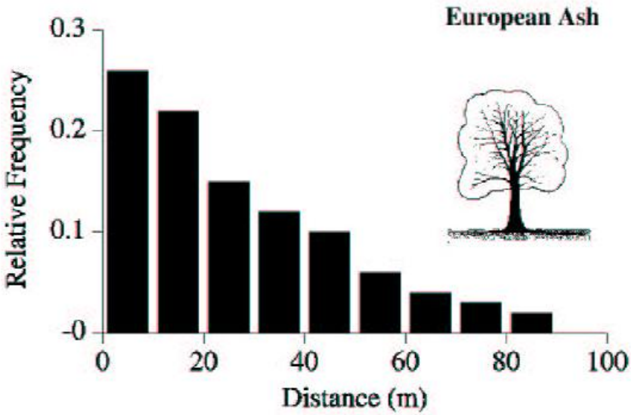
- Normal Distribution

$$k(x,y) = \sqrt{\frac{\alpha}{\pi}} \exp(-\alpha(x-y)^2)$$

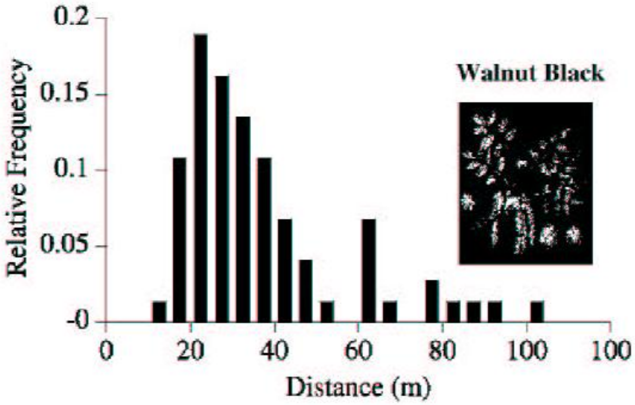
- Ballistic Dispersal

$$k(x,y) = \frac{3a}{2}\,|x-y|^{b-1} \exp(-a\,|x-y|^3)$$

Example 1


European Ash
Relative Frequency
0.3
0.2
0.1
-0
0
20
40
60
80
100
Distance (m)

Example 3


Walnut Black
Relative Frequency
0.2
0.15
0.1
0.05
-0
0
20
40
60
80
100
100
Distance (m)

Particular case -Harvesting

State Variable (Population)

$$N = N(\alpha) = (N_0(x), N_1(x), \cdots, N_T(x))$$

Control Variable (Harvesting Rate)

$$\alpha = (\alpha_0(x), \alpha_1(x), \cdots, \alpha_{T-1}(x))$$

Goal

Objective Functional (To Be Maximized)

$J(\alpha)$ defined as total revenue minus total cost in T time steps

Seek for an Optimal Control to Maximize Objective Functional

$$J(\alpha^*) = \max_{\alpha \in U} J(\alpha)$$

$$U = \left\{ \alpha \in \left(L^\infty(\Omega) \right)^T | 0 \le \alpha_t(x) \le M, t = 0, 1, \ldots, T-1 \right\}$$

for $M < 1$

Two Ways to Do Harvesting-Order

- Growth, Dispersal and Harvesting
- Growth, Harvesting and Dispersal

Growth, Dispersal and Harvesting

Linear growth & quadratic cost

$$N_{t+1}(x) = (1 - \alpha_t(x)) \int_\Omega k(x,y) r N_t(y) dy$$

$$J(\alpha) = \sum_{t=0}^{T-1} \int_\Omega e^{-\delta t} [A_t \alpha_t \int_\Omega k(x,y) r N_t(y) dy - \frac{B_t}{2} (\alpha_t(x))^2] dx$$

Hem Raj Joshi, Suzanne Lenhart, Holly Gaff
Optimal Control Applications and Methods, 2005

Growth, Harvesting and Dispersal (My Starting Problem)

Linear growth & Quadratic cost

$$N_{t+1}(x) = \int_\Omega k(x,y)(1-\alpha_t(y))rN_t(y)dy$$

$$J(\alpha) = \sum_{t=0}^{T-1} \int_\Omega e^{-\delta t}[A_t\alpha_t(y)rN_t(y) - \frac{B_t}{2}(\alpha_t(y))^2]dy$$

We Can Prove

- Existence of an Optimal Control
- Characterization of an Optimal Control
- Uniqueness of an Optimal Control

Adjoint and OC Characterization

Growth-Dispersal-Harvesting

Given an optimal control α^* and corresponding state solution $N^* = N(\alpha^*)$, there exists a weak solution $p \in (L^\infty(\Omega))^T$ satisfying the adjoint system:

$$\begin{aligned} p_{t-1}(x) =& r\int_\Omega (1-\alpha^*_{t-1}(y))p_t(y)k(y,x)dy + \\ & r\int_\Omega A_{t-1}e^{-\delta(t-1)}\alpha^*_{t-1}(y)k(y,x)dy \\ p_T(x) =& 0 \end{aligned}$$

where $t = T, \cdots, 2, 1$. Furthermore, for $t = 0, 1, 2, \cdots, T-1$,

$$\alpha^*_t(x) = min(max(\frac{(-e^{-\delta t}p_{t+1}(x) + A_t)\int_\Omega rk(x,y)N^*_t(y)dy}{B_t}, 0), M)$$

Adjoint and OC Characterization

Growth-Harvesting-Dispersal

Given an optimal control α^* and corresponding state solution $N^* = N(\alpha^*)$, there exists a weak solution $p \in (L^\infty(\Omega))^T$ satisfying the adjoint system:

$$p_{t-1}(x) = r(1-\alpha^*_{t-1}(x)) \int_\Omega p_t(y)k(y,x)dy + e^{-\delta t} r A_{t-1} \alpha^*_{t-1}(x)$$

$$p_T(x) = 0$$

where $t = T, \cdots, 2, 1$. Furthermore, for $t = 0, 1, 2, \cdots, T-1$,

$$\alpha^*_t(x) = min(max(\frac{[e^{-\delta t} \int_\Omega -p_{t+1}(y)k(y,x)dy + A_t] r N^*_t(x)}{B_t}, 0), M)$$

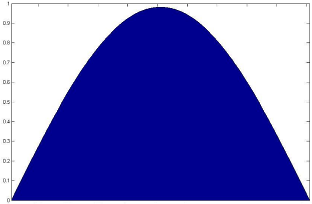
Numerical approach

Idea of algorithm

- Start with guess for controls and N_0
- Solve state equations for N forwards and adjoint equations for p backwards
- Update control with convex combination of old values and values from control
- Repeat until convergence of iterates

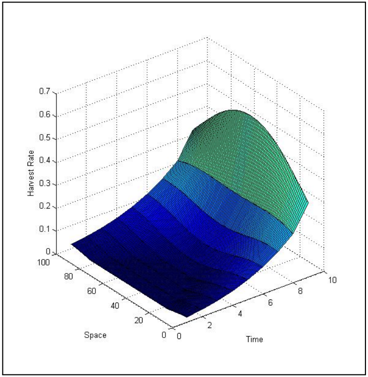
Kernel

$$k\left(x,y\right)=\begin{cases} 0, & \text{if } x\leq y-R \\ \dfrac{\pi}{4R}cos\left[\dfrac{\pi}{2R}\left|x-y\right|\right], & \text{if } y-R<x<y+R \\ 0, & \text{if } x\geq y+R \end{cases}$$

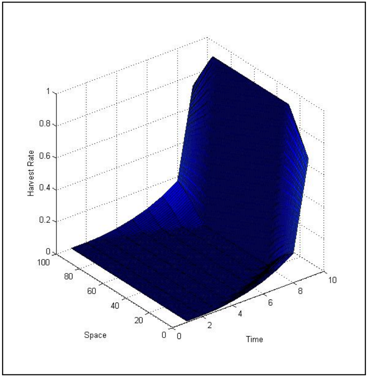


Optimal Harvesting Rates

Growth-Dispersal-Harvesting



Growth-Harvesting-Dispersal



Parameters

$$r = 1.8, B_t = 1000, \delta = 0.2, A_t = 1, R = 0.8$$

Future Work

- Contribute to other applications in life sciences, besides harvesting
  Such as forestry management, invasive species, fishery and disease spread
- Derive optimal control results for a more general framework

$$N_{t+1}(x) = h_t(\int_\Omega k(x,y) f(N_t(y), y) dy, \alpha_t(x))$$

or

$$N_{t+1}(x) = h_t(x, \int_\Omega k(x,y) f(N_t(y), y\alpha_t(y)) dy,)$$