# Artificial Viscosity Proper Orthogonal Decomposition

Zhu Wang

Department of Mathematics
Virginia Tech
wangzhu@vt.edu

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# Collaborators

Jeff Borggaard
Department of Mathematics
Interdisciplinary Center for Applied Mathematics
Virginia Tech
jborggaard@vt.edu

Traian Iliescu
Department of Mathematics
Interdisciplinary Center for Applied Mathematics
Virginia Tech
iliescu@vt.edu

# Overview

- Proper Orthogonal Decomposition (POD)
- POD Model
- Artificial Viscosity-POD (AV-POD) Model
- Theoretical results
- Numerical results
- Conclusions

# Proper Orthogonal Decomposition

Navier-Stokes equations

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\nabla P + \mathrm{Re}^{-1} \Delta \mathbf{U}$$
$$\nabla \cdot \mathbf{U} = 0$$

- $u^{(1)}, u^{(2)}, \cdots, u^{(m)}$ snapshots with rank d
- Find $\{\psi_1, ..., \psi_r\}$, $r <= d$, orthonormal basis

$$\min_{\{\psi_k\}_{k=1}^r} \frac{1}{m} \sum_{j=1}^{m} \| u^{(j)}(\cdot, t) - \sum_{k=1}^{r} (u^{(j)}(\cdot, t), \psi_k(\cdot))_H \psi_k(\cdot) \|_{L_2}^2$$

- $K\overrightarrow{v_k} = \lambda_k \overrightarrow{v_k}$, $\quad K_{ij} = \frac{1}{m}(u^{(j)}, u^{(i)})$
- $\psi_\ell = \frac{1}{\sqrt{\lambda_\ell}} \sum_{j=1}^{m} (\overrightarrow{v_\ell})_j \overrightarrow{u_j}$, $\ell = 1, \ldots, r$ [Kunisch & Volkwein, 2001]

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# POD model

- POD truncation

$$\mathbf{U} \approx u^r = \sum_{i=1}^{r} a_i(t)\psi_i(x)$$

- **POD-Galerkin model**

$$\left(\frac{\partial \mathbf{u}^r}{\partial t}, \psi_\mathbf{k}\right) + ((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \psi_\mathbf{k}) + \left(\frac{2}{Re}\mathbb{D}(\mathbf{u}^r), \nabla\psi_\mathbf{k}\right) = 0$$

- high Reynolds number $\Rightarrow \{\psi_{r+1}, \ldots, \psi_d\}$ important

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# Artificial Viscosity POD model

- **Energy cascade (Richardson , Kolmogorov)**
- **Artificial Viscosity $\Rightarrow$ Dissipation**
- **AV-POD model (Smagorinsky)**

$$\left(\frac{\partial \mathbf{u}^r}{\partial t}, \psi_{\mathbf{k}}\right) + ((\mathbf{u}^r \cdot \nabla)\mathbf{u}^r, \psi_{\mathbf{k}}) + \left(\left(\nu_S + \frac{2}{Re}\right) \mathbb{D}(\mathbf{u}^r), \nabla \psi_{\mathbf{k}}\right) = 0$$

$$\nu_S := C_s \, |\mathbb{D}(\mathbf{u}^r)|$$

$\nu_S$ models $\{\psi_{r+1}, \dots, \psi_d\}$

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$$\nu_S \text{ models } \{\psi_{r+1}, \dots, \psi_d\}$$

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# Theoretical results for Burgers Equation

Burgers equation

$$\begin{cases} u_t - \nu\, u_{xx} + u\, u_x = f & \text{in } \Omega \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \end{cases}$$

## Error Estimates

$$\frac{1}{m}\sum_{k=1}^{m} \|U_k - u(t_k)\|^2 \le C \left( \|u_0 - \mathrm{p}^{\ell} u_0\|^2 + \sum_{k=r+1}^{d} \lambda_k + \Delta t^2 \right)$$

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## Error Estimates

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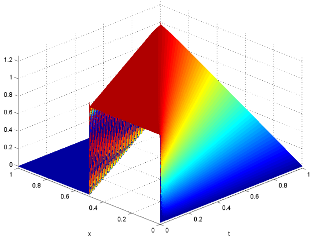
# Numerical results for Burgers Equation

## Experiment 1 [Kunisch & Volkwein, 1999]

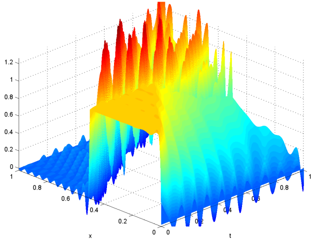
$$u_0(x) = \begin{cases} 1 & \text{if } x \in \left(0, \frac{1}{2}\right] \\ 0 & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases} .$$

- $u(0,t) = u(1,t) = 0$
- $f = 0$
- $[0, T] = [0, 1]$
- $\Delta x = 1/2048$;
- $\Delta t = 10^{-3}$;
- $\nu = 10^{-5}$;
- $m = 1000$ (snapshots).

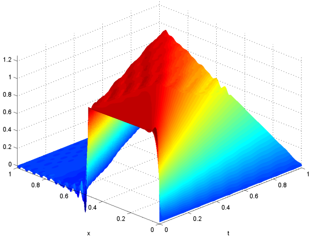
## DNS



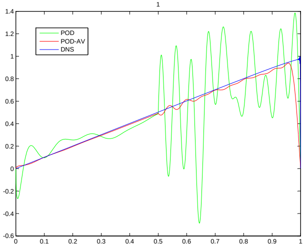

## POD model




## AV-POD model ($c = 1 \times 10^{-4}$)




## POD, AV-POD, DNS at $T = 1$



# Numerical results for Burgers Equation

## Experiment 2 [Mohseni, Zhao & Marsden, 2006]

$$u_0(x) = \frac{2}{\sqrt{\pi}} exp(-400(x - 1/8)^2);$$

- $u(0, t) = u(1, t) = 0$
- $f = 0$
- $[0, T] = [0, 1.5]$
- $\Delta x = 1/2048;$
- $\Delta t = 10^{-3};$
- $\nu = 3.75 \times 10^{-5};$
- $m = 1000$ (snapshots).

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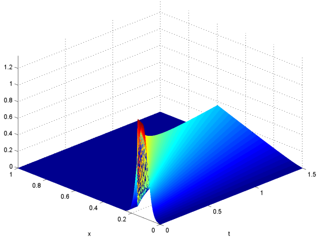
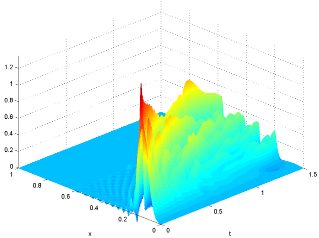
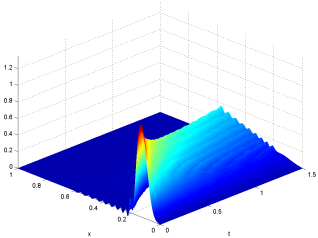
## DNS



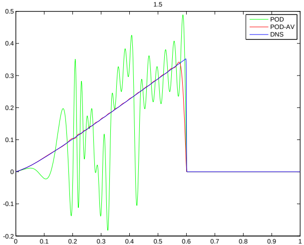

## POD model




## AV-POD model ($c = 5 \times 10^{-5}$)




## POD, AV-POD, DNS at $T = 1.5$



# Conclusions

## Accomplishments

- ◇ introduced AV-POD models
- ◇ theoretical error estimates (Burgers Equation)
- ◇ numerical experiments (Burgers Equation)
- ◇ *"Artificial Viscosity Proper Orthogonal Decomposition"*

*Borggaard, Iliescu,* **Wang**

## Future work

- ◇ 2D, 3D Navier-Stokes Equations
- ◇ turbulent pipe flow
- ◇ optimal model parameters
- ◇ improved models

# *Thank You!*

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