Exact controllability of the Schnakenberg model for Pattern Formation

Hoang Tran and Catalin Trenchea

hat25@pitt.edu

Department of Mathematics
University of Pittsburgh

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Pattern formation in morphogenesis



$$\begin{cases} u_t + \nabla \cdot (\mathbf{a}u) = \gamma(\alpha - u + u^2 v) + D_u \nabla^2 u \text{ in } Q \\ v_t + \nabla \cdot (\mathbf{a}v) = \gamma(\beta - u^2 v) + D_v \nabla^2 v \text{ in } Q \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu}(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \end{cases} \tag{1}$$

u, v : morphogen concentration.

α, β : source density.

The Controllability Problem

Given a state (u^*, v^*) and fixed time T,

Find α, β *such that* $u(T) = u^*, v(T) = v^*$,

where u, v are solutions of (1)?

Theorem 1 *Let* $(u^*, v^*) \in H^2(\Omega) \times H^2(\Omega)$ *be any steady-state solution to*

$$\begin{aligned} \nabla \cdot (\mathbf{a} u^*) &= \gamma(a^* - u^* + (u^*)^2 v^*) + D_u \nabla^2 u^* \text{ in } \Omega \\ \nabla \cdot (\mathbf{a} v^*) &= \gamma(b^* - (u^*)^2 v^*) + D_v \nabla^2 v^* \text{ in } \Omega \\ \frac{\partial u^*}{\partial \nu} &= \frac{\partial v^*}{\partial \nu} = 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

Then $\exists\, \delta > 0$ *such that* $\forall (u_0, v_0) \in H^1(\Omega) \times H^1(\Omega)$ *satisfying*

$$\|u^* - u_0\|_{\alpha_0(n)} + \|v^* - v_0\|_{H_0^1(\Omega)} \le \delta,$$

$\exists\, \alpha, \beta \in \mathbb{R}_+^*$ *such that*

$$u^{\alpha,\beta}(T) = u^*, v^{\alpha,\beta}(T) = v^*.$$

Let $y = u - u^*, z = v - v^*$.

Our problem now is to prove the **null controllability** for

$$\begin{aligned} y_t + \nabla \cdot (\mathbf{a} y) &= \gamma a + a_1 y + a_2 y^2 + a_3 z + a_4 yz + a_5 y^2 z + D_y \nabla^2 y \\ z_t + \nabla \cdot (\mathbf{a} z) &= \gamma b + b_1 y + b_2 y^2 + b_3 z + b_4 yz + b_5 y^2 z + D_z \nabla^2 z \end{aligned}$$

Initial condition: $y(0) = y_0 \equiv u_0 - u^*, z(0) = z_0 \equiv v_0 - v^*$

Linearized system:

$$\begin{aligned} y_t + \nabla \cdot (\mathbf{a} y) &= \gamma a + y(\underbrace{a_1 + a_2 \xi}_{=\mu_1}) + z(\underbrace{a_3 + a_4 \xi + a_5 \xi^2}_{=\mu_2}) + D_y \nabla^2 y \\ z_t + \nabla \cdot (\mathbf{a} z) &= \gamma b + y(\underbrace{b_1 + b_2 \xi}_{=\eta_1}) + z(\underbrace{b_3 + b_4 \xi + b_5 \xi^2}_{=\eta_2}) + D_z \nabla^2 z \end{aligned} \tag{3}$$

where $\xi \in L^\infty(Q)$.

Lemma 1 $\exists\, a, b > 0$ *such that* $y_\xi(T) = z_\xi(T) = 0$, *and*

$$a^2 + b^2 \leq \frac{|\Omega|}{\gamma^2} \left(\|y_0\|^2 + \|z_0\|^2 \right) e^{\int_0^T \Phi(t)dt} \tag{4}$$

where

$$\Phi(t) = \max\{\|\mu_1\|_{L^\infty(\Omega)} + \frac{\|\eta_1\|_\infty}{2} + \frac{\|\mu_2\|_\infty}{2}, \|\eta_2\|_\infty + \frac{\|\eta_1\|_\infty}{2} + \frac{\|\mu_2\|_\infty}{2}\}.$$

Proof of Theorem 1. We use Kakutani's fixed point theorem with

$K = \{\xi \in L^\infty(Q); \|\xi\|_\infty \leq R\}$
$\Phi(\xi) = \{y_\xi \in L^2(Q) |\, (y_\xi, z_\xi)$ solution of (3), $y_\xi(T) = z_\xi(T) = 0, a, b < M\}$

■

Proof of Lemma 1.

Consider the optimal control problem

$$\min_{a,b}\left\{\frac{a^2+b^2}{2}+\frac{1}{\varepsilon}\int_\Omega\left(\|y(T)\|^2+\|z(T)\|^2\right)dt\right\}$$

subject to (3)

which has a unique solution $(a_\varepsilon, b_\varepsilon)$ and $(y_\varepsilon, z_\varepsilon)$.

By the Pontryagin maximum principle we have

$$a_\varepsilon = \gamma \int_0^T \int_\Omega p_\varepsilon \, dx dt, \quad b_\varepsilon = \gamma \int_0^T \int_\Omega q_\varepsilon \, dx dt,$$

where $(p_\varepsilon, q_\varepsilon)$ is the solution to

$$\begin{aligned} -p_t - \mu_1 p - \eta_1 q + \mathbf{a} \cdot \nabla p - D_u \Delta p = 0, \\ -q_t - \mu_2 p - \eta_2 q + \mathbf{a} \cdot \nabla q - D_v \Delta q = 0, \end{aligned} \tag{5}$$

with final conditions

$$p(T) = -\frac{1}{\varepsilon} y_\varepsilon(T), \quad q(T) = -\frac{1}{\varepsilon} z_\varepsilon(T), \tag{6}$$

and boundary conditions

$$p\mathbf{a} \cdot \mathbf{n} - D_u \nabla p \cdot \mathbf{n} = 0, \quad q\mathbf{a} \cdot \mathbf{n} - D_v \nabla q \cdot \mathbf{n} = 0.$$

Then (3), (5) and (6) imply

$$\begin{aligned}
\gamma^2 \int_0^T\!\!\int_\Omega \left(p_\varepsilon^2 + q_\varepsilon^2\right) dx dt + \frac{1}{\varepsilon}\int_\Omega \left(y_\varepsilon^2(T) + z_\varepsilon^2(T)\right) dx \\
&= \int_\Omega \left(y_0 p_\varepsilon(0) + z_0 q_\varepsilon(0)\right) dx \\
&\le \frac{\gamma^2}{2C(T)}\int_\Omega \left(p_\varepsilon^2(0) + q_\varepsilon^2(0)\right) dx + \frac{C(T)}{2\gamma^2}\int_\Omega \left(y_0^2 + z_0^2\right) dx \\
&\le \frac{\gamma^2}{2}\int_0^T\!\!\int_\Omega \left(p_\varepsilon^2 + q_\varepsilon^2\right) dx dt + \frac{C(T)}{2\gamma^2}\int_\Omega \left(y_0^2 + z_0^2\right) dx
\end{aligned}$$

where

$$C(T) = \frac{1}{T} e^{\int_0^T \Phi(t) dt}.$$

Hence

$$\gamma^2 \int_0^T \int_\Omega \left(p_\varepsilon^2 + q_\varepsilon^2\right) dx dt + \frac{2}{\varepsilon} \int_\Omega \left(y_\varepsilon^2(T) + z_\varepsilon^2(T)\right) dx \leq \frac{C(T)}{\gamma^2} \int_\Omega \left(y_0^2 + z_0^2\right) dx$$

On a subsequence we have

$$a_\varepsilon \to a, \quad b_\varepsilon \to b \qquad \text{in } \mathbb{R},$$
$$y_\varepsilon \to y, \quad z_\varepsilon \to z \qquad \text{weakly in } L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T],L^2(\Omega))$$

where (y,z) satisfy (3) and $y(T) = 0, z(T) = 0$.

Moreover, a, b satisfy the estimate (4).

This completes the proof of Lemma 1. ∎

The exact controllability for other Turing-type reaction-diffusion systems when initial condition is close to the desired state

- Gierer-Meinhardt Model
- Thomas Model
- Chemotaxis Model

Can this method be applied successfully for the reaction-diffusion system with quite general condition?

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Thank you for your attention!