On the accuracy of the rotation form in simulations of the Navier-Stokes equations

Leo Rebholz

rebholz@clemson.edu

Department of Mathematical Sciences

Clemson University

Clemson, SC 29634

Outline

- Joint work with W. Layton, C. Manica, M. Neda and M. Olshanskii
- NSE and its nonlinearity
- Discrete nonlinearity forms: convective, skew-symmetric, rotational
- Examples that show drastic differences; rotational form bad!
- Discussion of possible cause
- Fix proposed: Grad-div stabilization
- Examples that show fix works

Navier-Stokes equations

- NSE in convective form

$$u_t + u \cdot \nabla u + \nabla p - Re^{-1} \Delta u = f, \;\; \nabla \cdot u = 0$$

- Vector identity:

$$u \cdot \nabla u = -u \times (\nabla \times u) + \frac{1}{2} \nabla |u|^2$$

- Using Bernoulli pressure $\tilde{p} = p + \frac{1}{2}|u|^2$, NSE in rotational form given by

$$u_t - u \times (\nabla \times u) + \nabla \tilde{p} - Re^{-1} \Delta u = f, \;\; \nabla \cdot u = 0$$

FEM scheme for NSE

- Naive Crank-Nicolson FEM scheme for NSE w/ convective form of NSE: $\forall (v_h, q_h) \in (X_h, Q_h)$

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h) + (u_h^{n+1/2} \cdot \nabla u_h^{n+1/2}, v_h)$$

$$-(p_h^{n+1/2}, \nabla \cdot v_h) + Re^{-1}(\nabla u_h^{n+1/2}, \nabla v_h) = (f(t^{n+1/2}), v_h)$$

$$(\nabla \cdot u_h^{n+1}, q_h) = 0$$

- This scheme known to be unstable

Variations of NSE scheme

- For stability, scheme often altered
- Skew symmetric form:

$$(u_h \cdot \nabla u_h, v_h) \rightarrow \frac{1}{2}(u_h \cdot \nabla u_h, v_h) - \frac{1}{2}(u_h \cdot \nabla v_h, u_h)$$

- Rotational form:

$$(u_h \cdot \nabla u_h, v_h) - (p_h, \nabla \cdot v_h) \rightarrow$$
$$- (u_h \times (\nabla \times u_h), v_h) - (\tilde{p}_h, \nabla \cdot v_h)$$

- In continuous case all 3 forms are equivalent

Variations of NSE scheme

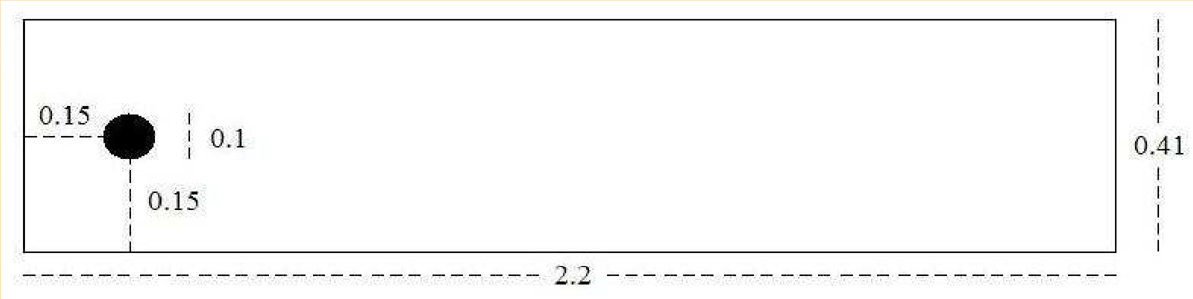
- For both skew symmetric and rotational form schemes, we can prove

$$\|u_h^M\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \|\nabla u_h^{n+1/2}\|^2 \leq C(Re, f, T, u_0)$$

- So both schemes are stable and conserve energy
- There are situations where one may want to use the rotational form scheme instead of convective form
  - Conservation of helicity
  - More robust preconditioners for large Reynolds numbers available
  - However, Rotational form can give bad results

Skew-symmetric vs. Rotational

Channel Flow around a cylinder benchmark problem
Domain:



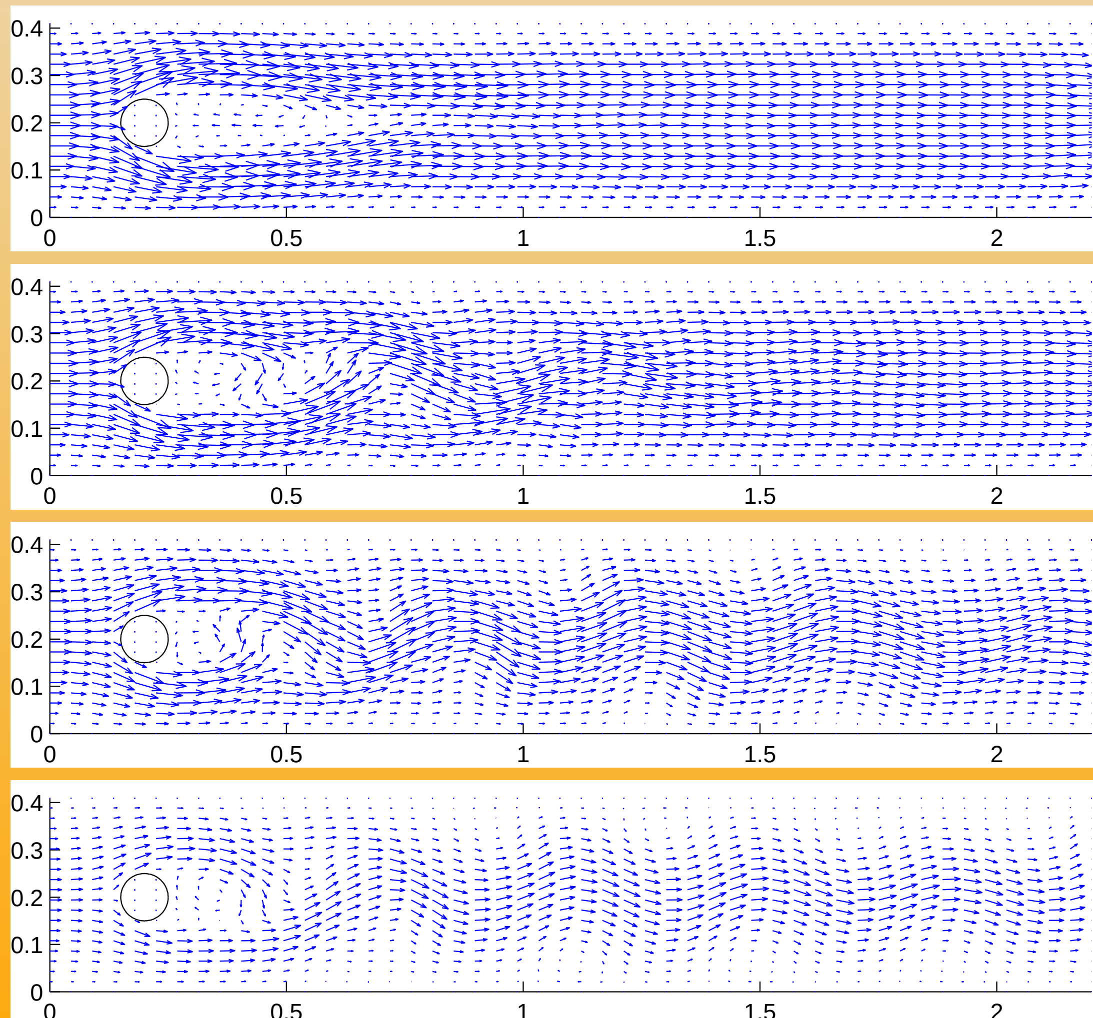
The time dependent inflow and outflow profile are

$$u_1(0, y, t) = u_1(2.2, y, t) = \frac{6}{0.41^2} \sin(\pi t/8) y(0.41 - y)$$
$$u_2(0, y, t) = u_2(2.2, y, t) = 0.$$

No slip boundaries, 0 initial condition, $0 \leq Re(t) \leq 100$.

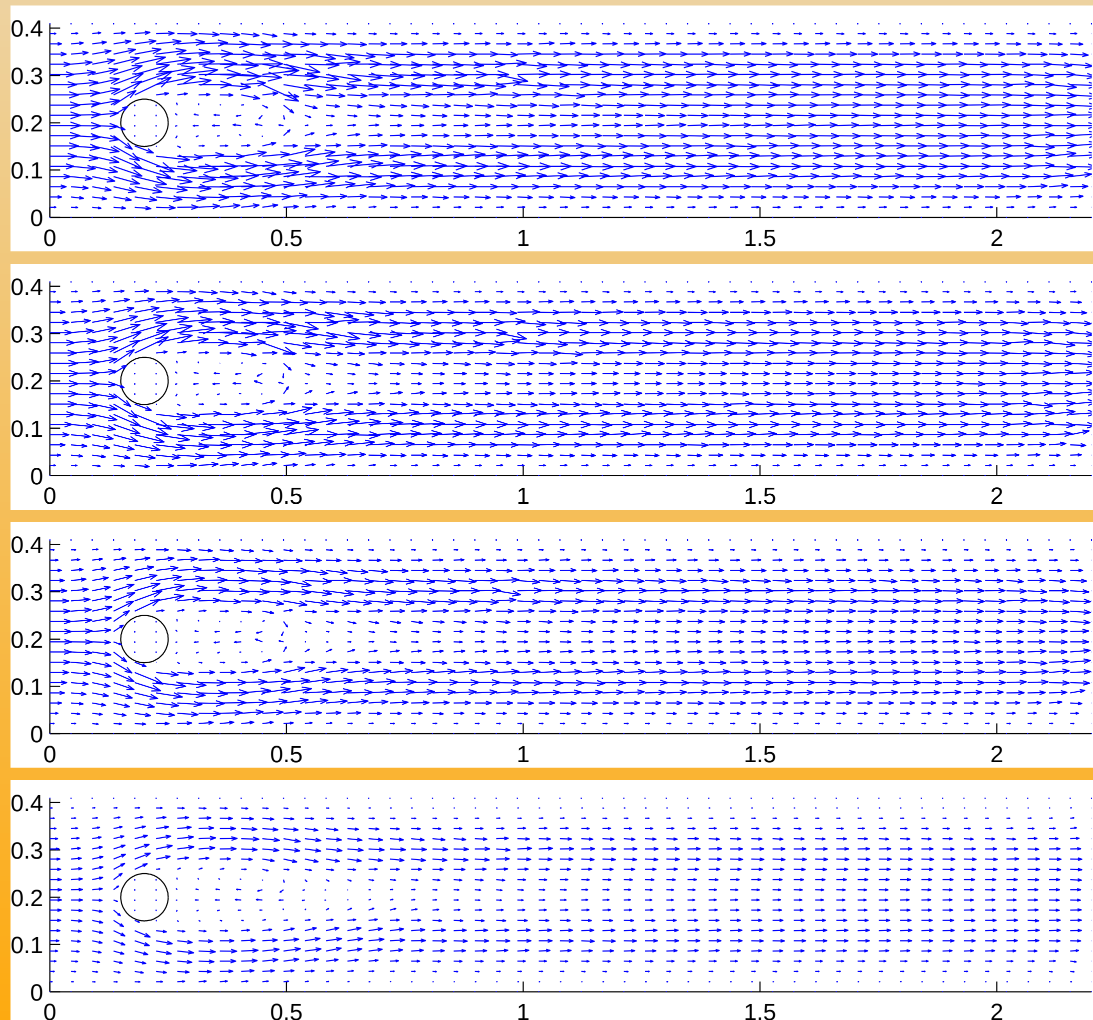
Skew-symmetric vs. Rotational

From (P_2, P_1) skew-symm scheme at t=4,5,6,7 (Good!)



Skew-symmetric vs. Rotational

(P_2, P_1) rotational form, same mesh, t=4,5,6,7 (Bad!)



Skew-symmetric vs. Rotational

- How can we get such different answers?
  - Increased pressure error for rotational form
  - $\tilde{p} = p + \frac{1}{2}|u|^2$ more complex than p
    - Boundary layers in Bernoulli pressure
  - On meshes where p_h is resolved, $\tilde{p}_h$ may not be
  - For (P_k, P_{k-1}) elements, "some of u" gets approximated with P_{k-1} for Bernoulli pressure
- Roughly speaking, if this pressure term is dominant, velocity error scales with Re * pressure error
- Similar problem in Stokes equations can be fixed with grad-div stabilization (Olshanskii/Reusken)
- Analysis of pressure term same in Stokes and NSE...

Fixing Rotational Form

- Consider Stokes problem, where there is same pressure term on RHS of velocity error equation

$$\begin{aligned} -Re^{-1}\Delta u + \nabla p &= f, \;\; \nabla \cdot u = 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

- FEM scheme: $\forall v_h \in X_h, \; q_h \in Q_h$, solve

$$\begin{aligned} Re^{-1}(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) + (q_h, \nabla \cdot u_h) &= (f, v_h) \\ (\nabla \cdot u_h, q_h) &= 0 \end{aligned}$$

Fixing Rotational Form

- Typical velocity error bound for scheme:

$$\|\nabla(u-u_h)\| \leq C(\inf_{v\in X_h}\|\nabla(u-v_h)\| + Re\inf_{q_h\in Q_h}\|p_h-p\|)$$
$$\leq Ch^2(\|\nabla\nabla u\| + Re*\|\nabla p\|)$$

- Olshanskii / Reusken: Add grad-div stabilization, $\gamma(\nabla\cdot u_h,\nabla\cdot v_h)$ to LHS of FEM scheme for Stokes equations, with $\gamma=O(1)$.
- New error equation:

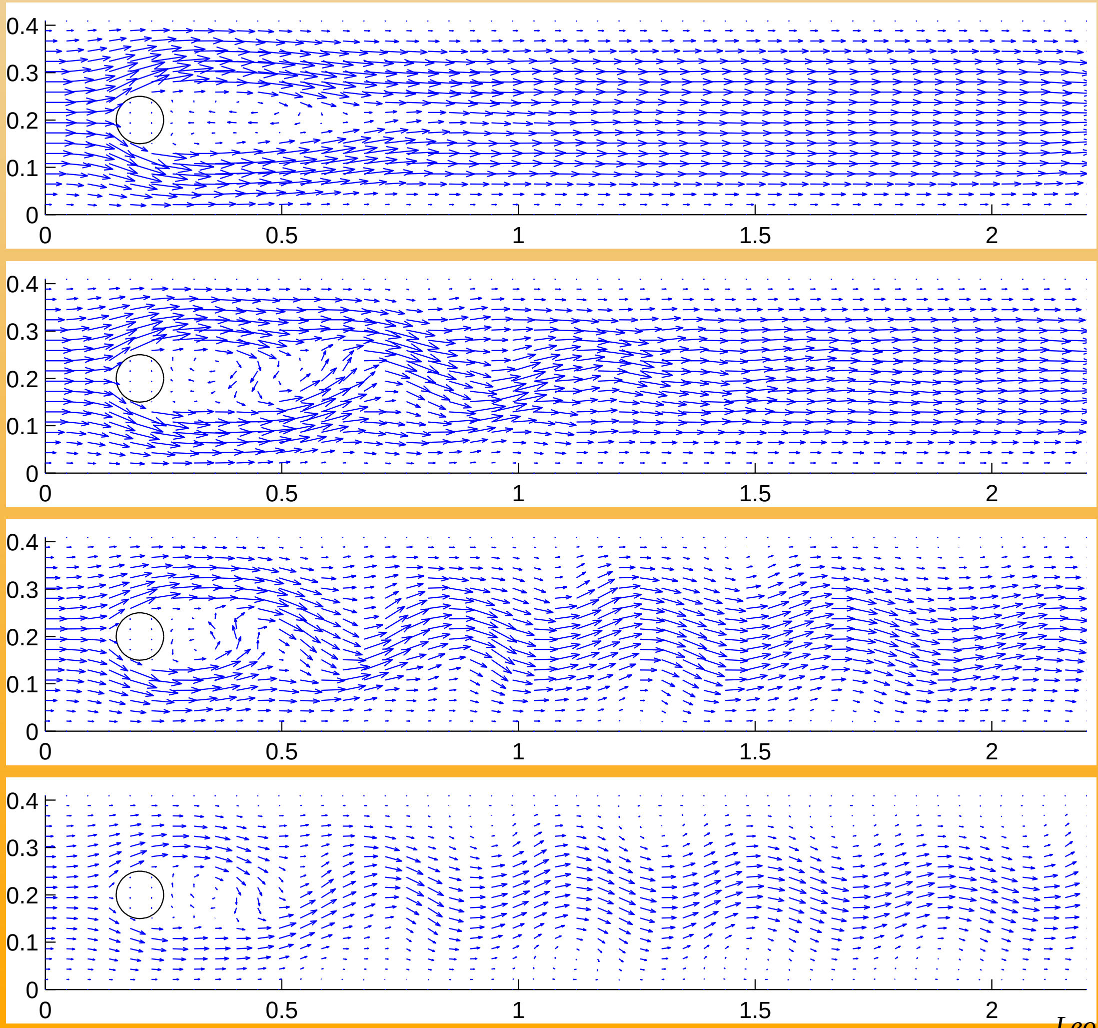
$$\|\nabla(u-u_h)\| \leq Ch^2\sqrt{Re}(\|\nabla\nabla u\| + \|\nabla p\|)$$

Fixing Rotational Form

- In short, the grad div term lets you handle $\inf_{q_h \in Q_h} \|p - q_h\| \|\nabla \cdot (u_h - U)\|$ in a different way
- Without stabilization, you have to "hide" the $\|\nabla \cdot (u_h - U)\|$ part on the left hand side in the $Re^{-1}\|\nabla(u_h - U)\|^2$ term
  - Creates a $Re * \inf_{q_h \in Q_h} \|p - q_h\|^2$ on the RHS
- With stabilization, will have a $\|\nabla \cdot (u_h - U)\|^2$ on the LHS, and so only a $\inf_{q_h \in Q_h} \|p - q_h\|^2$ will be left on the RHS
- Dependence of velocity error on pressure error is greatly reduced.

Fixing Rotational Form

From (P_2, P_1) grad-div stabilized rotational form at t=4,5,6,7 (Good!)

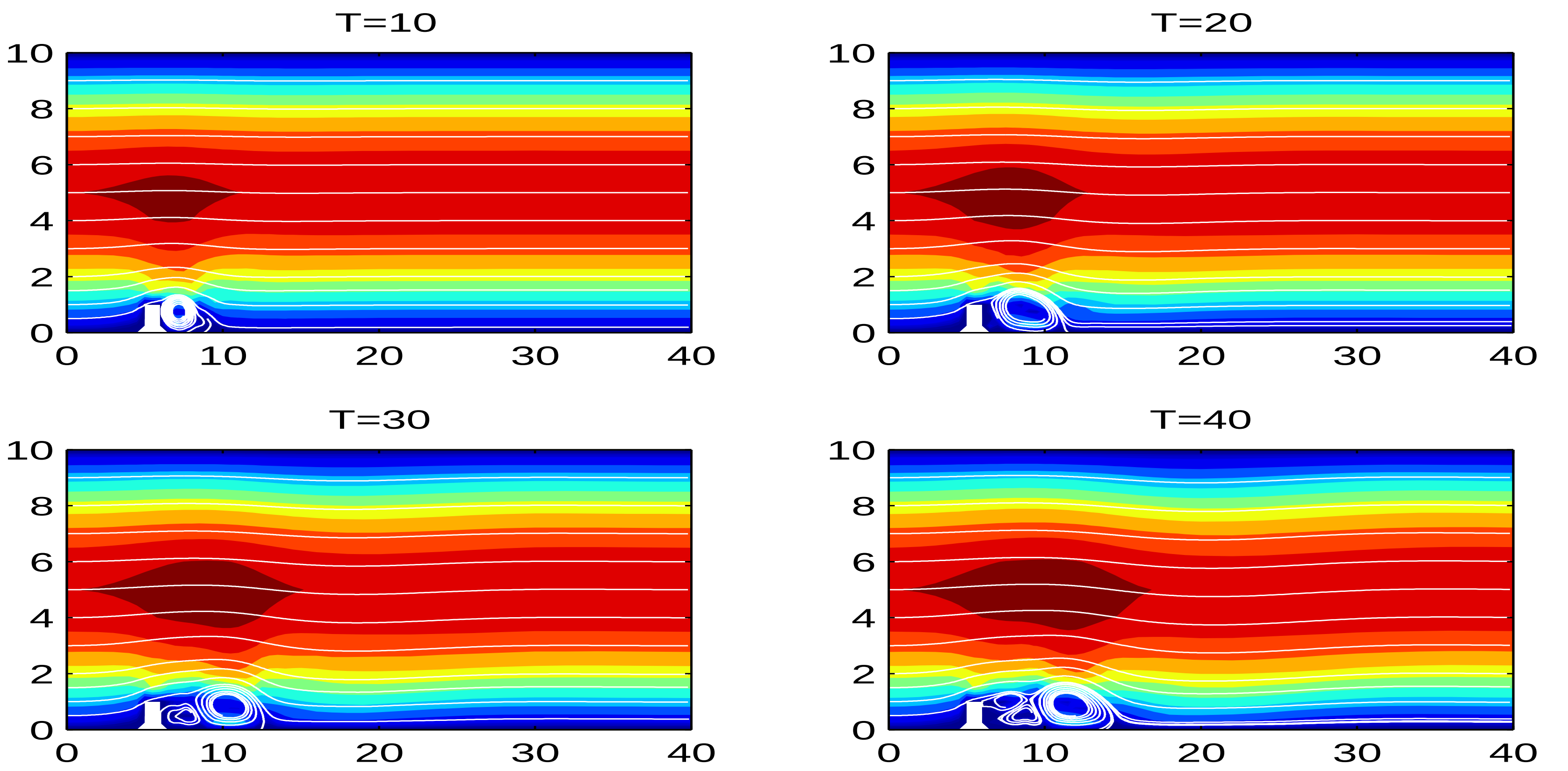


Flow over a Step

- Second test problem: Benchmark channel flow over a step
- FreeFem, (P_2, P_1) elements
- Re=600
- Dirichlet parabolic inflow, do-nothing outflow
- No slip sides
- Eddies should form behind step, detach, and new ones should form

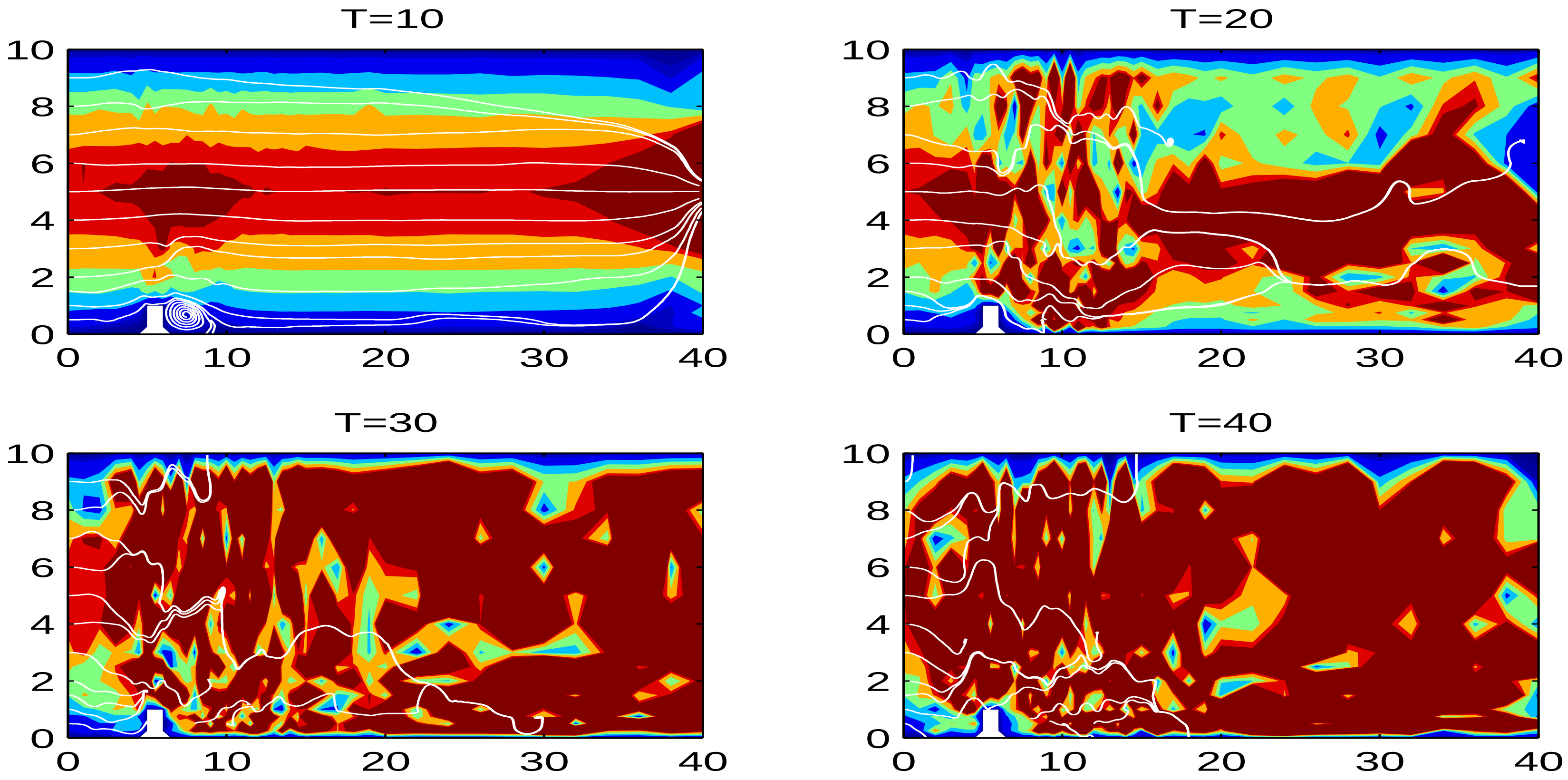
Flow over a Step

NSE with skew-symmetric form of nonlinearity



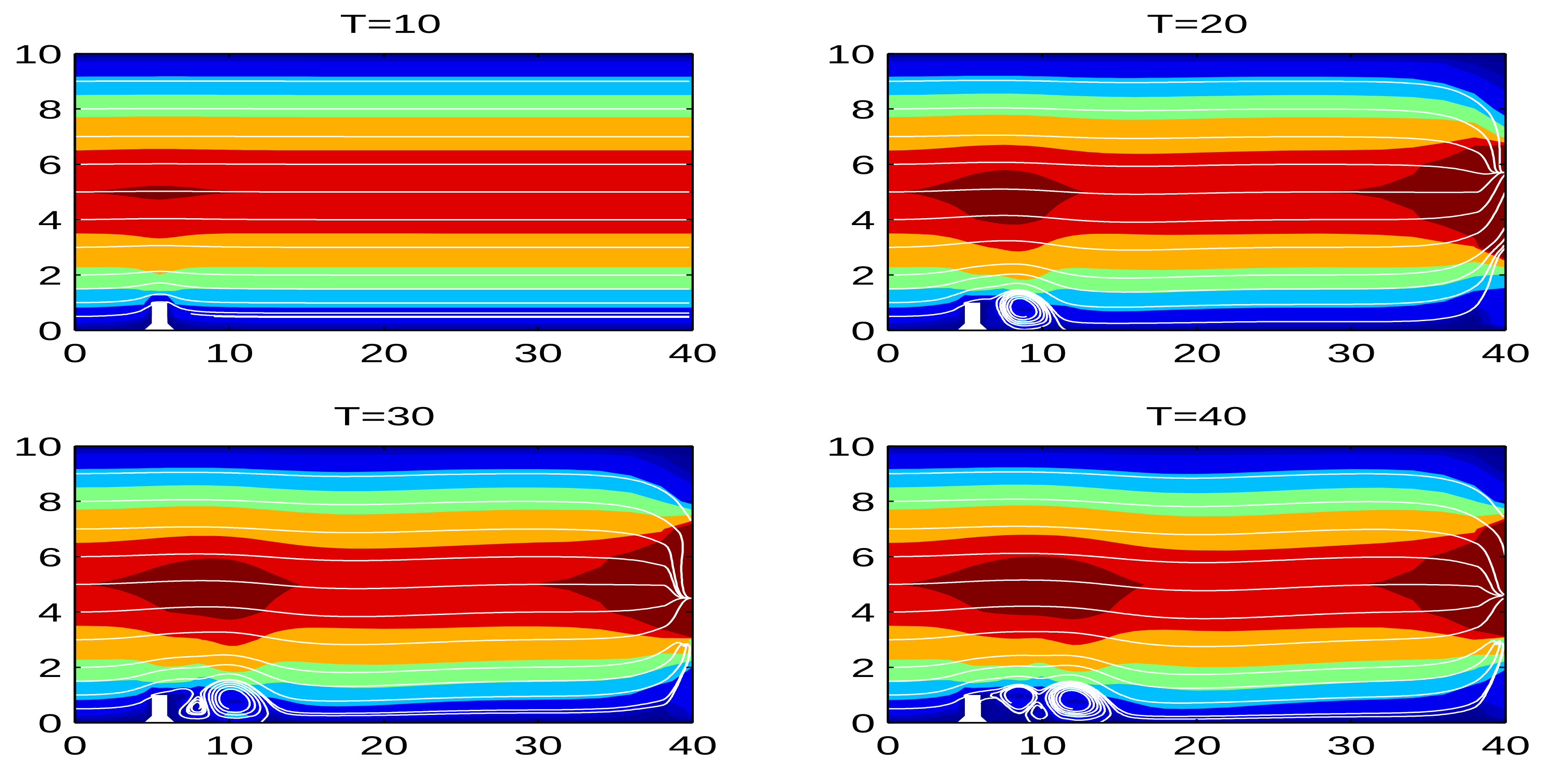
Flow over a Step

NSE w/ rotational form of nonlinearity



Flow over a Step

NSE w/ rotational form and grad-div stabilization; unresolved issue w/ what is correct outflow bdry condition



Conclusions

- There are cases where skew-symmetric FEM scheme tends to be much more accurate than rotational form scheme
- Difference is due to increased error in Bernoulli pressure, which in turn increases velocity error
- Adding grad-div stabilization can lessen the effect, especially for higher Re, thus reducing error in velocity field
- We are NOT claiming rotation + grad-div is best, but if you want to use rotational form, adding grad-div can greatly improve accuracy.