Optimal Filtering with Mobile Sensors

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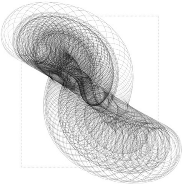

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The natural state space for the problem is $L^2(\Omega)$ and the domain of the differential operator in the right hand side is $H^2(\Omega) \cap H_0^1(\Omega)$.

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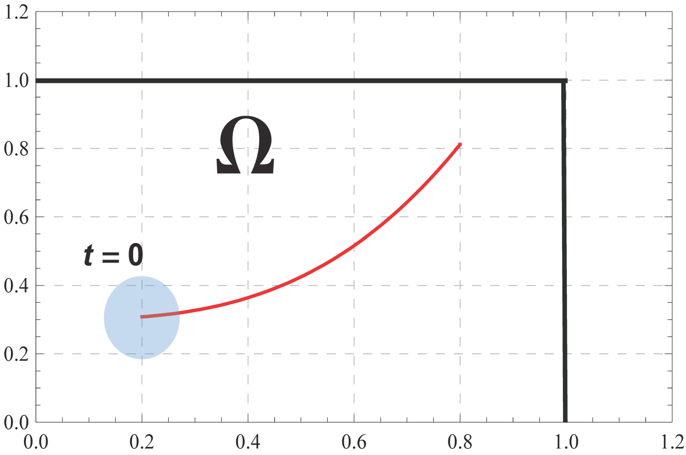
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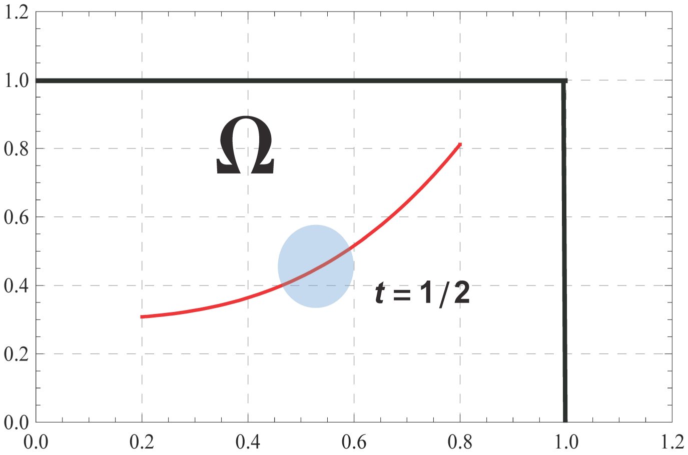
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Ω
t = 0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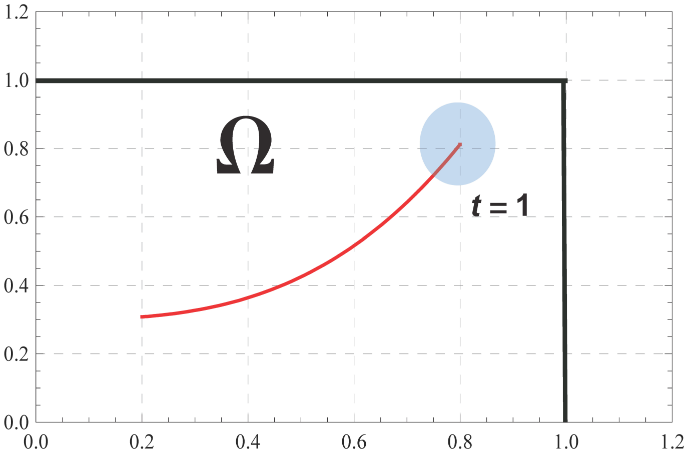
0.8
1.0
1.2
Ω
t = 1/2


1.2
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0.8
0.6
0.4
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Ω
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$$h(t, \mathbf{x}) = \chi(\mathbf{x}, \hat{\mathbf{x}}(t)) \int_{\Omega} \chi(\mathbf{y}, \hat{\mathbf{x}}(t)) K(t, \mathbf{y}) T(t, \mathbf{y}) \, \mathrm{d}\mathbf{y} + \text{noise},$$

where $\chi(\mathbf{x}, \mathbf{y}) = 1$ if $\|\mathbf{x} - \mathbf{y}\| \leq \delta$ and zero everywhere else.

Then, we may assume that the measurements (not only the ones along trajectories) are explicitly determined by

$$h(t,\mathbf{x}) = \int_\Omega k(t,\mathbf{x},\mathbf{y})\,T(t,\mathbf{y})\,\mathrm{d}\mathbf{y} + \nu(t),$$

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3) Sensors with point wise limitations, i.e., $|k(t,\mathbf{x},\mathbf{y})| < 1$ for all $\mathbf{x},\mathbf{y} \in \Omega$ and $t \in [0,1]$

The general form for possible measurements

In order to make the measurements realistic, we should enforce that for each $t \in [0,1]$,

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This implies that for each $t \in [0,1]$ the operator that represents the measurement is Hilbert-Schmidt since the integral representation with the square integrable kernel is a concrete realization of $\mathscr{I}_2$ over $L^2(\Omega)$.

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with measured output

$$w(t) = C(t)z(t) + \nu(t),$$

where, for each $t \in [0, 1]$, $C(t)$ belongs to a family $\mathcal{F}$ of Hilbert - Schmidt operators and $\mathcal{F}$ is determined by the nature of the measurements (stationary, along trajectories, etc...).

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Question: HOW ARE WE GOING TO CHOOSE $C(t)$?

If we construct a Kalman filter with the measured output $w(t) = C(t)z(t) + \nu(t)$, then the covariance operator $\Sigma(t)$ between the real state $z(t)$ and the estimated one $\hat{z}(t)$ is the mild solution of the Riccati differential equation

$$\dot{\Sigma} = A(t)\Sigma + \Sigma A(t) + B(t)B^{*}(t) - \Sigma C^{*}(t)C(t)\Sigma,$$

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Answer: Then, as an analogy with the finite dimensional case, we would like minimize

$$J(C) = \int_0^1 \mathrm{Tr}(\Sigma(t; C))\, \mathrm{d}t,$$

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$$J(C) = \int_0^1 \mathrm{Tr}(\Sigma(t;C))\,\mathrm{d}t,$$

with $C = C(t)$...but we don't know if $\Sigma(t)$ is a trace class operator!!!!

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1. Since $B(t)$ and $C(t)$ are Hilbert-Schmidt, then if Σ_0 is trace class, then $\Sigma(t)$ is trace class for every t.

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1. Since $B(t)$ and $C(t)$ are Hilbert-Schmidt, then if Σ_0 is trace class, then $\Sigma(t)$ is trace class for every t.
2. Even more, if the mappings $t \mapsto B^*(t)B(t)$ and $t \mapsto C^*(t)C(t)$ are trace norm continuous, then also is the mapping $t \mapsto \Sigma(t)$ and Σ solves an integral version of the Riccati equation (but not strongly, uniformly in trace norm!)

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$$J(C) = \int_0^1 \mathrm{Tr}(\Sigma(t; C))\, \mathrm{d}t,$$

where the family of operators $C(t)$ belong some family $\mathcal{F}$ of the Hilbert Schmidt operators for each $t \in [0, 1]$.

Under all our previous assumptions...

There are minimizers...

If the kernel of the family $\{C(t)\}$ of integral operators is uniformly bounded, i.e., $|k(t,x,y)| \leq M$, then there are minimizers $\hat{C}(t)$,

$$J(\hat{C}) = \inf_{C \in \mathcal{F}} J(C),$$

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1. $\mathcal{F}$ is the family of stationary sensors.
2. $\mathcal{F}$ is the family of time dependent sensors.
3. $\mathcal{F}$ is the family of sensors with fixed range δ that move along trajectories (under several restrictive hypothesis).

Consider the one dimensional convection diffusion

$$T_t = \epsilon T_{xx} - aT_x + b(x)\eta(t),$$

on $0 \leq t \leq 0.2$, and $0 \leq x \leq 1$. With $T_x(t,0) = T_x(t,1) = 0$ and $T(0,x) = T_0(x)$.

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Suppose that the family of sensors $\mathcal{F}$, correspond to those which move uniformly in time, from $x_0 \in [0,1]$ to $x_1 \in [0,1]$ and with range $\delta = 0.05$.

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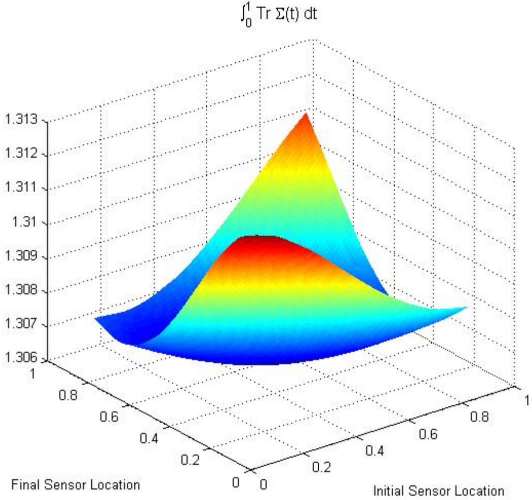
Then, we can parameterize $J(C) = \int_0^1 \mathrm{Tr}(\Sigma)\,\mathrm{d}t$, with x_0 and x_1 as $J(x_0, x_1)$...

Finite element approximation with $n = 128$



Then, it appears that we have to move the sensor uniformly along $x_0 + x_1 \simeq 1$ to minimize the functional.

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Apparently the minimum is attained when $x_0 \simeq 0.592$ and $x_1 \simeq 0.590...$ which is more or less stationary.

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3. The minimization problem has a solution over a wide range of dynamic sensors...

The main goal for this project is:

- Try to prove that there are minimizers when the trajectories are prescribed by $\dot{x} = f(t, \mathbf{x}, u)$ and u belongs to some family of admissible controls (preferably $|u(t)| \leq 1$).

The main goal for this project is:

- Try to prove that there are minimizers when the trajectories are prescribed by $\dot{x} = f(t, \mathbf{x}, u)$ and u belongs to some family of admissible controls (preferably $|u(t)| \leq 1$).
- Develop approximation schemes to compute numerical solutions for the problem.

THANK YOU!