Optimal Filtering with Mobile Sensors

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February 21, 2009



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 $\begin{array}{c} \mbox{Problem statement} \\ \mbox{Trace ideal properties of } \Sigma \mbox{ and consequences} \\ \mbox{Some pictures and issues} \\ \mbox{Conclusions} \\ \mbox{Conclusions} \end{array}$

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The system

Possible measurements Abstract statement of the problem What is our criteria?

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Two dimensional parabolic problem

Consider the **convection-diffusion** process in $\Omega = [0,1] \times [0,1]$ and in $t \in [0,1]$

$$rac{\partial}{\partial t}T = (c^2\Delta + \mathbf{a}(x,y)\cdot
abla)T + B(t)\eta(t);$$

with η a Wiener process,

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with η a Wiener process, for each $t \in [0, 1]$ the operator B(t) is a little bit more than compact (Hilbert-Schmidt)

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with η a Wiener process, for each $t \in [0, 1]$ the operator B(t) is a little bit more than compact (Hilbert-Schmidt) and boundary and initial conditions

$$T(t,x,y)\Big|_{\partial\Omega} = 0,$$
 $T(0,x,y) = T_0(x).$

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with η a Wiener process, for each $t \in [0, 1]$ the operator B(t) is a little bit more than compact (Hilbert-Schmidt) and boundary and initial conditions

$$T(t,x,y)\Big|_{\partial\Omega} = 0,$$
 $T(0,x,y) = T_0(x).$

The natural state space for the problem is $L^2(\Omega)$ and the domain of the differential operator in the right hand side is $H^2(\Omega) \cap H^1_0(\Omega)$.

Suppose that we can only "measure" $T(t, \mathbf{x})$ on some smooth trajectory $\hat{\mathbf{x}}(t)$, inside Ω .

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Suppose that we can only "measure" $T(t, \mathbf{x})$ on some smooth trajectory $\hat{\mathbf{x}}(t)$, inside Ω . So, we may assume that the sensor measures an average value of $T(t, \mathbf{x})$ within a fixed range δ of the position of the sensor for each t = [0, 1]

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So, these type of measurements $h(t, \mathbf{x})$ have the form,

 $\begin{array}{c} \textbf{Problem statement} \\ \text{Trace ideal properties of } \Sigma \text{ and consequences} \\ \text{Some pictures and issues} \\ \text{Conclusions} \end{array}$

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So, these type of measurements $h(t, \mathbf{x})$ have the form,

$$h(t, \mathbf{x}) = \chi(\mathbf{x}, \hat{\mathbf{x}}(t)) \int_{\Omega} \chi(\mathbf{y}, \hat{\mathbf{x}}(t)) T(t, \mathbf{y}) \, \mathrm{d}\mathbf{y} + \mathsf{noise},$$

 $\begin{array}{c} \label{eq:problem_statement} \\ \mbox{Trace ideal properties of } \Sigma \mbox{ and consequences} \\ \mbox{Some pictures and issues} \\ \mbox{Conclusions} \end{array}$

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where $\chi(\mathbf{x}, \mathbf{y}) = 1$ if $\|\mathbf{x} - \mathbf{y}\| \le \delta$ and zero everywhere else...

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where $\chi(\mathbf{x}, \mathbf{y}) = 1$ if $\|\mathbf{x} - \mathbf{y}\| \le \delta$ and zero everywhere else...but we can generalize this a little bit more...

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$$h(t, \mathbf{x}) = \chi(\mathbf{x}, \hat{\mathbf{x}}(t)) \int_{\Omega} \chi(\mathbf{y}, \hat{\mathbf{x}}(t)) \mathcal{K}(t, \mathbf{y}) \mathcal{T}(t, \mathbf{y}) \, \mathrm{d}\mathbf{y} + \text{noise},$$

where $\chi(\mathbf{x}, \mathbf{y}) = 1$ if $\|\mathbf{x} - \mathbf{y}\| \le \delta$ and zero everywhere else.

Then, we may assume that the measurements (not only the ones along trajectories) are explicitly determined by

$$h(t, \mathbf{x}) = \int_{\Omega} k(t, \mathbf{x}, \mathbf{y}) T(t, \mathbf{y}) \,\mathrm{d}\mathbf{y} +
u(t),$$

for some other Wiener process ν (that is uncorrelated with η), and the possible kernels k correspond some admissible family.

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for some other Wiener process ν (that is uncorrelated with η), and the possible kernels k correspond some admissible family. By changing the admissible set of kernels, we may deal with

1) Stationary sensors, i.e., $k(t, \mathbf{x}, \mathbf{y}) = k(\mathbf{x}, \mathbf{y})$.

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1) Stationary sensors, i.e., $k(t, \mathbf{x}, \mathbf{y}) = k(\mathbf{x}, \mathbf{y})$.

2) Sensors with limited energy, i.e.,

$$\sup_{t\in[0,1]}\|k(t,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}<1.$$

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2) Sensors with limited energy, i.e.,

$$\sup_{t\in[0,1]}\|k(t,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}<1.$$

3) Sensors with point wise limitations, i.e., $|k(t, \mathbf{x}, \mathbf{y})| < 1$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $t \in [0, 1]$

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The general form for possible measurements

In order to make the measurements realistic, we should enforce that for each $t \in [0, 1]$,

 $\|k(t,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)} < \infty.$

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The general form for possible measurements

In order to make the measurements realistic, we should enforce that for each $t \in [0,1]$,

 $\|k(t,\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}<\infty.$

This implies that for each $t \in [0, 1]$ the operator that represents the measurement is Hilbert-Schmidt since the integral representation with the square integrable kernel is a concrete realization of \mathscr{I}_2 over $L^2(\Omega)$.

Then, we can rewrite the problem as an abstract infinite dimensional model of the form

$$\dot{z}(t) = A(t)z(t) + B(t)\eta(t) \in L^2(\Omega),$$

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Then, we can rewrite the problem as an abstract infinite dimensional model of the form

$$\dot{z}(t) = A(t)z(t) + B(t)\eta(t) \in L^2(\Omega),$$

with measured output

$$w(t) = C(t)z(t) + \nu(t),$$

where, for each $t \in [0, 1]$, C(t) belongs to a family \mathcal{F} of Hilbert -Schmidt operators and \mathcal{F} is determined by the nature of the measurements (stationary, along trajectories, etc...).

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where, for each $t \in [0, 1]$, C(t) belongs to a family \mathcal{F} of Hilbert -Schmidt operators and \mathcal{F} is determined by the nature of the measurements (stationary, along trajectories, etc...). **Question:** HOW ARE WE GOING TO CHOOSE C(t)?

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If we construct a Kalman filter with the measured output $w(t) = C(t)z(t) + \nu(t)$, then the covariance operator $\Sigma(t)$ between the real state z(t) and the estimated one $\hat{z}(t)$ is the mild solution of the Riccati differential equation

$$\dot{\Sigma} = A(t)\Sigma + \Sigma A(t) + B(t)B^*(t) - \Sigma C^*(t)C(t)\Sigma,$$

with some $\Sigma(0) = \Sigma_0$, then...

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with some $\Sigma(0) = \Sigma_0$, then...

Answer: Then, as an analogy with the finite dimensional case, we would like minimize

$$J(C) = \int_0^1 \operatorname{Tr}(\Sigma(t; C)) \, \mathrm{d}t,$$

with C = C(t)...

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with some $\Sigma(0) = \Sigma_0$, then...

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$$J(C) = \int_0^1 \operatorname{Tr}(\Sigma(t; C)) \, \mathrm{d}t,$$

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with C = C(t)...but we don't know if $\Sigma(t)$ is a trace class operator!!!!

A needed Theorem Control problem on the Riccati equation The main results

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Theorem (PROPERTIES OF THE MAP $t \mapsto \Sigma(t)$)

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Theorem (PROPERTIES OF THE MAP $t \mapsto \Sigma(t)$)

Since B(t) and C(t) are Hilbert-Schmidt, then if Σ₀ is trace class, then Σ(t) is trace class for every t.

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Theorem (PROPERTIES OF THE MAP $t \mapsto \Sigma(t)$)

- Since B(t) and C(t) are Hilbert-Schmidt, then if Σ₀ is trace class, then Σ(t) is trace class for every t.
- Even more, if the mappings t → B*(t)B(t) and t → C*(t)C(t) are trace norm continuous, then also is the mapping t → Σ(t) and Σ solves an integral version of the Riccati equation (but not strongly, uniformly in trace norm!)

A needed Theorem Control problem on the Riccati equation The main results

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Finally our problem is the following:

A needed Theorem Control problem on the Riccati equation The main results

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Finally our problem is the following: Minimize (if possible) the functional

$$J(C) = \int_0^1 \operatorname{Tr}(\Sigma(t; C)) \, \mathrm{d}t,$$

where the family of operators C(t) belong some family \mathcal{F} of the Hilbert Schmidt operators for each $t \in [0, 1]$.

A needed Theorem Control problem on the Riccati equation The main results

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Under all our previous assumptions...

There are minimizers...

If the kernel of the family $\{C(t)\}$ of integral operators is uniformly bounded, i.e., $|k(t, x, y)| \leq M$, then there are minimizers $\hat{C}(t)$,

$$J(\hat{C}) = \inf_{C \in \mathcal{F}} J(C),$$

where $\hat{C} \in \mathcal{F}$, if...

A needed Theorem Control problem on the Riccati equation The main results

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 $\bullet \ \mathcal{F} \text{ is the family of stationary sensors.}$

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 $\bullet \ \mathcal{F} \text{ is the family of stationary sensors.}$

2 \mathcal{F} is the family of time dependent sensors.

A needed Theorem Control problem on the Riccati equation The main results

Under all our previous assumptions...

There are minimizers...

If the kernel of the family $\{C(t)\}$ of integral operators is uniformly bounded, i.e., $|k(t, x, y)| \leq M$, then there are minimizers $\hat{C}(t)$,

$$J(\hat{C}) = \inf_{C \in \mathcal{F}} J(C),$$

where $\hat{C} \in \mathcal{F}$, if...

- $\bullet \ \mathcal{F} \text{ is the family of stationary sensors.}$
- **2** \mathcal{F} is the family of time dependent sensors.
- *F* is the family of sensors with fixed range δ that move along trajectories (under several restrictive hypothesis).

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Consider the one dimensional convection diffusion

$$T_t = \epsilon T_{xx} - aT_x + b(x)\eta(t),$$

on $0 \le t \le 0.2$, and $0 \le x \le 1$. With $T_x(t,0) = T_x(t,1) = 0$ and $T(0,x) = T_0(x)$.

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Consider the one dimensional convection diffusion

$$T_t = \epsilon T_{xx} - aT_x + b(x)\eta(t),$$

on $0 \le t \le 0.2$, and $0 \le x \le 1$. With $T_x(t,0) = T_x(t,1) = 0$ and $T(0,x) = T_0(x)$. Suppose that the family of sensors \mathcal{F} , correspond to those which move uniformly in time, from $x_0 \in [0,1]$ to $x_1 \in [0,1]$ and with range $\delta = 0.05$.

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Consider the one dimensional convection diffusion

$$T_t = \epsilon T_{xx} - aT_x + b(x)\eta(t),$$

on $0 \le t \le 0.2$, and $0 \le x \le 1$. With $T_x(t,0) = T_x(t,1) = 0$ and $T(0,x) = T_0(x)$. Suppose that the family of sensors \mathcal{F} , correspond to those which move uniformly in time, from $x_0 \in [0,1]$ to $x_1 \in [0,1]$ and with range $\delta = 0.05$. Then, we can parameterize $J(C) = \int_0^1 \operatorname{Tr}(\Sigma) \, \mathrm{d}t$, with x_0 and x_1 as $J(x_0, x_1)$...

An example by E. M. Cliff

Finite element approximation with n = 128



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Then, it appears that we have to move the sensor uniformly along $x_0 + x_1 \simeq 1$ to minimize the functional.

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Then, it appears that we have to move the sensor uniformly along $x_0 + x_1 \simeq 1$ to minimize the functional. Apparently the minimum is attained when $x_0 \simeq 0.592$ and $x_1 \simeq 0.590$... which is more or less stationary.

We've proved that... But the main goal is...

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• The solution of the Riccati equation is trace class continuous and it is unique.

We've proved that... But the main goal is...

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- The solution of the Riccati equation is trace class continuous and it is unique.
- The minimization problem has a solution over the stationary sensors.

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- The solution of the Riccati equation is trace class continuous and it is unique.
- The minimization problem has a solution over the stationary sensors.
- The minimization problem has a solution over a wide range of dynamic sensors...

Problem statement Trace ideal properties of Σ and consequences Some pictures and issues **Conclusions**

We've proved that... But the main goal is...

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The main goal for this project is:

• Try to prove that there are minimizers when the trajectories are prescribed by $\dot{x} = f(t, \mathbf{x}, u)$ and u belongs to some family of admissible controls (preferably $|u(t)| \leq 1$).

We've proved that... But the main goal is...

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The main goal for this project is:

- Try to prove that there are minimizers when the trajectories are prescribed by $\dot{x} = f(t, \mathbf{x}, u)$ and u belongs to some family of admissible controls (preferably $|u(t)| \le 1$).
- Develop approximation schemes to compute numerical solutions for the problem.

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We've proved that... But the main goal is...

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THANK YOU!

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