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Lighthill acoustic analogy

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Fluid turbulent flows tend to generate noise. This differs from sound produced by the vibration of solids. There's an interest in prediction of the noise in the following areas :

- Ground transportation such as cars and trains.
- Aircraft and jet planes. The fighter jets that are being designed would produce about 148 decibels while 150 damage internal organs of pilots.
- Medicine of blood flows. Measuring sound from blood flowing through a valve of the heart.
- Submarine detection.
- Consumer industry.

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Ω_1 : turbulent flow generating the sound
Ω/Ω_1 : the acoustic wave propagation in the unperturbed media
The goal is to estimate sound intensity in Ω.

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- The main problem is coupling 1 and 2
- The final purpose is finding p' and sound intensity $\mathbf{I} = p^{'}\mathbf{u}$

- The compressible NSE in Ω:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \nabla \cdot \mathbb{S} + \rho \mathbf{f}$$

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- Mathematical consequence :

$$-\Delta p = \nabla \cdot (\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbb{S} - \rho \mathbf{f}) - \frac{\partial^2 \rho}{\partial t^2}$$

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- Equivalently but only in unperturbed media :

$$\frac{1}{a_0^2} \frac{\partial^2 p^{'}}{\partial t^2} - \Delta p = \nabla \cdot (\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbb{S} - \rho \mathbf{f}) + \frac{\partial^2}{\partial t^2} (\frac{p^{'}}{a_0^2} - \rho)$$

In Ω/Ω_1 this results in

$$\frac{1}{a_0^2}\frac{\partial^2 p^{'}}{\partial t^2} - \Delta p^{'} = 0$$

Lighthill's idea was to extend fluctuations $p^{'}$ and $\rho^{'}$ to the turbulent region. Lighthill analogy describes the acoustic wave propagation via the equation

$$\frac{1}{a_0^2}\frac{\partial^2 p^{'}}{\partial t^2} - \Delta p^{'} = \nabla \cdot (\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbb{S} - \rho \mathbf{f})$$

To solve this wave equation it's necessary to know the RHS which contains the information about the turbulent flow.

- Solve the incompressible NSE in Ω_1 using Finite Element Method on the mesh of size h_1. The spaces of $\mathbf{u}_{h_1}$ and p_{h_1} must satisfy the inf-sup condition.

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- Solve the Lighthill analogy in the whole Ω with various boundary conditions
- The numerical error consists of the error coming from FEM approximation of the Lighthill analogy and the error from computing the RHS of the analogy.

Lemma

If $\nabla \cdot \mathbf{u} = 0$ *then* $\nabla \cdot \nabla \cdot \mathbb{S} = 0$

■ Since $\nabla \cdot \mathbb{S} = \mu \Delta \mathbf{u}$ then $\nabla \cdot \nabla \cdot \mathbb{S} = \mu \sum_{i=0}^{3} \frac{\partial^2}{\partial x_i^2} (\nabla \cdot \mathbf{u}) = 0$
▲

Lemma

If $\rho \equiv \rho_0$ *and* $\nabla \cdot \mathbf{u} = 0$ *then* $\nabla \cdot \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$

■ $\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 \mathbf{u} \cdot \nabla \mathbf{u}$
So $\nabla \cdot \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho_0 (u_i u_{j,i})_{,j} = \rho_0 u_{i,j} u_{j,i} = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t$ ▲
Thus

$$\nabla \cdot (\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbb{S} - \rho \mathbf{f}) = \rho_0 \nabla \mathbf{u} : \nabla \mathbf{u}^t - \rho_0 \nabla \cdot \mathbf{f}$$

Let $Q(u, v) := \rho_0 \nabla \mathbf{u} : \nabla \mathbf{v}^t$

An example with non-reflecting boundary conditions :

$$\frac{\partial^2 p'}{\partial t^2} - a_0^2 \Delta p' = a_0^2 \cdot (Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0 \cdot \nabla \cdot \mathbf{f}) \ \forall (t, x) \in (0, T) \times \Omega_1,$$

$$\frac{\partial^2 p'}{\partial t^2} - a_0^2 \Delta p' = 0 \ \forall (t, x) \in (0, T) \times \Omega / \Omega_1,$$

$$p'(0, x) = q_1(x), \ \frac{\partial p'}{\partial t}(0, x) = q_2(x) \ \forall x \in \Omega$$

$$\nabla p' \cdot \mathbf{n} + \frac{1}{a_0} \frac{\partial p'}{\partial t} = 0 \ \forall (t, x) \in (0, T) \times \partial \Omega$$

The variational formulation for this case is :
find $p^{'} \in L^2(0, T; H^1(\Omega))$ such that $\frac{\partial p^{'}}{\partial t} \in L^2(0, T; H^1(\Omega))$, $\frac{\partial^2 p^{'}}{\partial t^2} \in L^2(0, T; L^2(\Omega))$ and

$$\left(\frac{\partial^2 p^{'}}{\partial t^2}, v\right) + a_0^2 \left(\nabla p^{'}, \nabla v\right) + a_0 \left\langle \frac{\partial p^{'}}{\partial t}, v \right\rangle = \\ = a_0^2 (Q(\mathbf{u}, \mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f}, v)_{\Omega_1}$$

$$\forall v \in H^1(\Omega), 0 < t < T,$$

$$(p^{'}(0, \cdot), v) = (q_1(\cdot), v) \ \forall v \in H^1(\Omega), \tag{1}$$

$$\left(\frac{\partial p^{'}}{\partial t}(0, \cdot), v\right) = (q_2(\cdot), v) \ \forall v \in H^1(\Omega). \tag{2}$$

$S^k(\Omega)$ is the finite dimensional space of continuous piecewise polynomials of degree no more than $k-1$. FEM approximation is based on the formulation : find a twice differentiable map $p_h^{'} : [0, T] \to S^k(\Omega)$ such that

$$\left(\frac{\partial^2 p_h^{'}}{\partial t^2}, v_h\right) + a_0^2 \left(\nabla p_h^{'}, \nabla v_h\right) + a_0 \left\langle \frac{\partial p_h^{'}}{\partial t}, v_h \right\rangle = $$

$$= a_0^2 (Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - \rho_0 \nabla \cdot \mathbf{f}, v_h)_{\Omega_1}$$

$$\forall v_h \in S^k(\Omega), 0 < t < T$$

$$p_h^{'}(0, \cdot) \text{ approximates } q_1(\cdot) \text{ in } S^k(\Omega), \tag{3}$$

$$\frac{\partial p_h^{'}}{\partial t}(0, \cdot) \text{ approximates } q_2(\cdot) \text{ in } S^k(\Omega). \tag{4}$$

Typically, the derivation of L^2-estimates is based on the energy method and was done by Dupont in 1973 and with some improvement on regularity by Baker in 1976. In our case the RHS is perturbed and requires more analysis.

Theorem

The FEM solution is stable.

$$\left\|\frac{\partial p_h^{'}}{\partial t}\right\|^2 + a_0^2 \|\nabla p_h^{'}\|^2 \leqslant$$

$$C \left(a_0^4 \int_0^t \|Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) \; -\rho_0 \nabla \cdot \mathbf{f}\|^2 d\tau + \left\|\frac{\partial p_h}{\partial t}(0, \cdot)\right\|^2 + a_0^2 \|\nabla p_h(0, \cdot)\|^2 \right)$$

Define H^1-projection $\tilde{p}$ of the solution p' by the formula

$$a_0^2(\nabla p', \nabla v_h) + (p', v_h) = a_0^2(\nabla \tilde{p}, \nabla v_h) + (\tilde{p}, v_h) \ \forall v_h \in S^k(\Omega)$$

Theorem

Let the variational solution p' satisfy conditions : $p', \frac{\partial p'}{\partial t} \in L^\infty(H^k(\Omega))$ and $\frac{\partial^2 p'}{\partial t^2} \in L^2(H^k(\Omega))$. If the initial conditions are taken so that $\|(p_h' - \tilde{p})(0,\cdot)\|_{H^1(\Omega)} + \left\|\frac{\partial}{\partial t}(p_h' - \tilde{p})(0,\cdot)\right\| \leqslant C_1 h^k$ with some posititve constant C_1 independent of h, then the solution p_h' satisfies :

$$\|p' - p_h'\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(p' - p_h')\right\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant C\left(h^k + \|Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1}) - Q(\mathbf{u}, \mathbf{u})\|_{L^2(L^2(\Omega_1))}\right)$$

Calling $\psi = p_h^{'} - \tilde{p}$, $\eta = \tilde{p} - p^{'}$ and using energy method, one can obtain the following inequality :

$$\frac{d}{dt}\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2 + a_0^2\|\nabla\psi\|^2\right) + 2\left|\sqrt{a_0}\cdot\frac{\partial \psi}{\partial t}\right|^2_{L^2(\partial\Omega)} \leqslant$$

$$C\left(\left\|\frac{\partial \psi}{\partial t}\right\|^2 + \|\psi\|^2 + \|\eta\|^2 + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2\right) + 2a_0\left|\left\langle\frac{\partial \eta}{\partial t}, \frac{\psi}{\partial t}\right\rangle\right| +$$

$$+ a_0^2\|Q_{h_1} - Q\|^2$$

Integrating in time and using standard inequalities with Gronwall's inequality, it's possible to obtain :

$$\left\|\frac{\partial\psi}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))} + \|\psi\|^2_{L^\infty(H^1(\Omega))} + \left\|\sqrt{a}\cdot\frac{\partial\psi}{\partial t}\right\|^2_{L^2(L^2(\partial\Omega))} \leqslant$$
$$C[\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))} + \|\eta\|^2_{L^2(L^2(\Omega))} + \left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} +$$
$$+\left\|\frac{\partial\psi}{\partial t}(0,\cdot)\right\|^2 + \|\psi(0,\cdot)\|^2_{H^1(\Omega)} + \int_0^t \|Q_{h_1} - Q\|^2 d\tau],$$

where $C = C(T)$ grows exponentially fast.

Lemma

Let $p^{'}, \frac{\partial p^{'}}{\partial t} \in L^{\infty}(H^k(\Omega))$ and $\frac{\partial^2 p^{'}}{\partial t^2} \in L^2(H^k(\Omega))$. Then for some constant C independent of h

$$\left\| \frac{\partial^r \eta}{\partial t^r} \right\|_{L^s(H^k(\Omega))} + \left\| \frac{\partial^r \eta}{\partial t^r} \right\|_{L^s(H^{-\frac{1}{2}}(\Omega))} \leqslant Ch^k,$$

where $s = \infty, \infty, 2$ for $r = 0, 1, 2$ respectively.

The theorem follows from the previous lemma and the previous inequality.

The optimal estimate of the RHS $L^2(L^2(\Omega_1))$-error is still an issue. Using inverse inequalities, we can end up with

$$\int_0^t \|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\|^2 d\tau \leqslant$$
$$Ch_1^{-\frac{n}{2}} \cdot \int_0^t (h_1^{2m-2}\|\partial^m \mathbf{u}\|^2_{L^4(\Omega_1)} + \|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|^2) d\tau,$$

where $n = 2$ or $n = 3$ is the dimension of the physical space. Depending on which finite elements are used, the rate of convergence for $\|\nabla(\mathbf{u}-\mathbf{u}_{h_1})\|^2$ may be obtained in the form $O(h^s)$. But the regularity condition $\mathbf{u} \in L^\infty(0,T;W^{1,4}(\Omega_1)) \cap L^2(0,T;W^{m,4}(\Omega_1))$ is required in this case.

The error analysis for negative norms has been done in case of the homogeneous Neumann boundary condition $\nabla p^{'} \cdot \mathbf{n} = 0$.
Consider the solution operator T.

$$Tf = u \text{ solves } \begin{cases} -a_0^2 \Delta u + u = f, \Omega \\ \nabla u \cdot \mathbf{n} = 0, \partial\Omega \end{cases}$$

This operator is self-adjoint and positive definite and thus generates an inner product and a norm by formulas :

$$(u, v)_{-1} = (Tu, v), \|u\|_{-1} = \sqrt{(Tu, u)}$$

If all the conditions of the main theorem are satisfied, then the error estimate in negative norms is given by the inequality

$$\left\|\frac{\partial}{\partial t}(p^{'} - p_h^{'})\right\|_{L^\infty(H^{-1}(\Omega))} + \|p^{'} - p_h^{'}\|_{L^\infty(L^2(\Omega))} \leqslant$$

$$C\left(h^{k+1} + h\|Q(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}_{h_1},\mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))} + \right.$$

$$\left. + \frac{1}{h}\|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))} + \left\|\frac{\partial}{\partial t}(p^{'} - p_h^{'})\right\|_{-1}(0) + \|p^{'} - p_h^{'}\|(0)\right)$$

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  Either use a straightforward definition of intensity or use duality approach. The last implies that we formulate a dual problem to the given wave equation and make error analysis for it.
- The Lighthill analogy for low Mach numbers also may be written as

$$\frac{1}{a_0^2}\frac{\partial^2 p^{'}}{\partial t^2} - \Delta p^{'} = -\Delta p$$

  Pressure p comes from the incompressible NSE in the turbulent region Ω_1.

- Analysis for different types of boundary conditions.

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- Research of the LES. Derivation of the avereged Lighthill's analogy. Theoretical analysis and numerical schemes.

Thanks for your attention !