Optimal Control of Discrete Time Models

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Outline - Two Models

- Background of Optimal Control of Discrete Systems
- Background of Cardiopulmonary Resuscitation Model
- Assumptions and format of the model
- Optimal control formulation and some results
- Background of the Epidemic Model of Rabies in Raccoons.
- Formulation of the model
- Numerical Results

Optimal Control of Discrete Systems

Goal: Adjust a control (coefficient or source term) in a system to achieve a desired goal.
Pontryagin's Maximum Principle handles optimal control of systems of ODEs.
Extension to discrete time models

Control Problem

Given a control $u = (u_0, u_1, \ldots, u_{T-1})$ and initial state x_0, the state equation is given by the
difference equation

$$x_{k+1} = g(x_k, u_k, k)$$

for $k = 0, 1, 2, \ldots, T-1$. Note that the state has one more component than the control

$$x = (x_0, x_1, \ldots, x_T),$$

Goal

$$J(u) = \phi(x_T) + \sum_{k=0}^{T-1} f(x_k, u_k, k)$$

Hamiltonian

$$H_k = f(x_k, u_k, k) + \lambda_{k+1} g(x_k, u_k, k), \quad \text{for } k = 0, 1, \ldots, T-1.$$

Notice the indexing on the adjoint.
Necessary conditions

$$\lambda_k = \frac{\partial H_k}{\partial x_k}$$
$$\lambda_T = \phi'(x_T^*)$$
$$\frac{\partial H_k}{\partial u_k} = 0 \text{ at } u^*.$$

Motivation - CPR work

Each year, more than 250,000 people die from cardiac arrest in the USA alone. Despite widespread use of cardiopulmonary resuscitation, the survival of patients recovering from cardiac arrest remains poor.

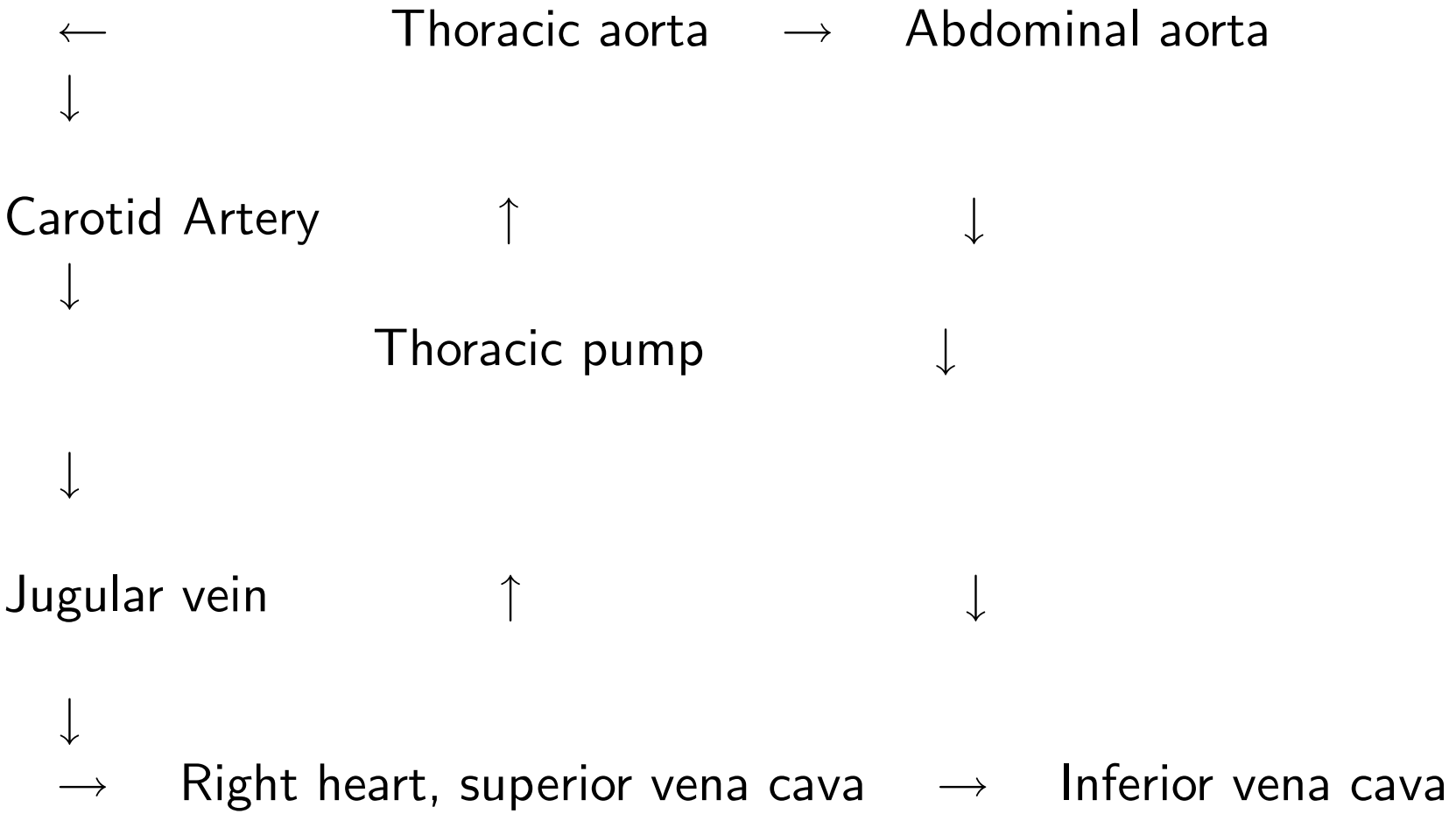
The rate of survival for CPR performed out of the hospital is 3%, while for patients who have cardiac arrest in the hospital, the rate of survival is 10-15%.

The standard and various alternative CPR techniques such as interposed abdominal compression IAC, active compression-decompression, and Lifestick CPR have been represented in various models. Here, we consider a model for CPR allowing chest and abdomen compression and decompression.

We apply the optimal control strategy for improving resuscitation rates to a validated circulation model developed by Babbs.

In his model, heart and blood vessels are represented as resistance-capacitive networks, pressures in the chest and in the vascular components as voltages, blood flow as electric current, and cardiac and venous valves as diodes (devices w/ flow in only 1 direction).
Reference: Babbs, Circulation 1999.

Diagram of Circulation Model



State Variables

As controls, we choose the the pattern of the external pressure on the chest and on the abdomen. The pressure state variables are as follows:

P_1 pressure in abdominal aorta

P_2 pressure in inferior vena aorta

P_3 pressure in carotid artery

P_4 pressure in jugular vein

P_5 pressure in thoracic aorta

P_6 pressure in rt. heart,superior vena cava

P_7 pressure in thoracic pump and left heart.

The chosen CPR model consists of seven difference equations, with time as the discrete underlying variable.
At the step n, when time is $n\Delta t$, the pressure vector is denoted by:

$$P(n) = (P_1(n), P_2(n), ..., P_7(n)).$$

We assume that the initial pressure values are known, when n $= 0$.
To make the chest pressure profiles medically reasonable, assume i.e., $u_i(0) = u_i(N-1)$.

$$u_1 = (u_1(0), u_1(1), ..., u_1(N-2), u_1(0)),$$
$$u_2 = (u_2(0), u_2(1), ..., u_2(N-2), u_2(0)),$$

Difference Equations Model for $n = 1, 2, ..., N-1$ (in vector notation)

$$P(1) = P(0) + T_1(u_1(0)) + T_2(u_2(0)) + \Delta t F(P(0)), \quad (1)$$

$$P(n+1) = P(n) + T_1(u_1(n) - u_1(n-1)) \quad (2)$$

$$+ T_2(u_2(n) - u_2(n-1)) + \Delta t F(P(n)), \quad (3)$$

$$T_1(u_1(n)) = (0, 0, 0, 0, t_p u_1(n), t_p u_1(n), u_1(n)),$$

$$T_2(u_2(n)) = (u_2(n), u_2(n), 0, 0, 0, 0, 0).$$

Note that the pressure vector depends on the control, $P = P(u)$, and the calculation of the pressures at the next time step requires the values of the controls at the current and previous time steps. We use extension of the discrete version of Pontryagin's Maximum Principle.

Show function $F(P(n))$ by some of its seven components:

$$\frac{1}{c_{aa}}\left[\frac{1}{R_a}(P_5(n)-P_1(n))-\frac{1}{R_s}(P_1(n)-P_2(n))\right]$$

$$\frac{1}{c_{ivc}}\left[\frac{1}{R_s}(P_1(n)-P_2(n))-\frac{1}{R_v}(P_2(n)-P_6(n)\right]$$

$$\frac{1}{c_{car}}\left[\frac{1}{R_c}(P_5(n)-P_3(n))-\frac{1}{R_h}(P_3(n)-P_4(n))\right]$$

continued

$$\frac{1}{c_{jug}}\left[\frac{1}{R_h}(P_3(n)-P_4(n))-\frac{1}{R_j}V(P_4(n)-P_6(n))\right]$$

$$\frac{1}{c_{ao}}\left[\frac{1}{R_o}V(P_7(n)-P_5(n))-\frac{1}{R_c}(P_5(n)-P_3(n))\right.$$

$$\left.+\frac{1}{R_a}(P_5(n)-P_1(n))-\frac{1}{R_{ht}}V(P_5(n)-P_6(n))\right]$$

where the valve function is defined by
$V(s) = s$ if $s \geq 0$
$V(s) = 0$ if $s \leq 0$.

Three valves: between compartments 4 - 6 AND 5 - 7 AND 5 - 6.

Goal

Choose the control set $U \subset \Re^{2N}$, defined as:

$$U = \{(u_1, u_2) | u_i(0) = u_i(N-1)$$

$$-K_i \leq u_i(n) \leq L_i, i = 1, 2, n = 0, 1, \ldots, N-2\}.$$

We define the objective functional $J(u_1, u_2)$ to be maximized

$$\sum_{n=1}^{N} [P_5(n) - P_6(n)] - \sum_{n=0}^{N-2} [\frac{B_1}{2} u_1^2(n) + \frac{B_2}{2} u_2^2(n)] \qquad (4)$$

Adjoint and OC Characterization

Given an optimal control $(u_1^* u_2^*)$ and the corresponding state solution, $P^* = P(u_1^*, u_2^*)$, there exists a solution of the adjoint system:

$$\lambda(n-1) \tag{5}$$

$$= \lambda(n) + \Delta t M^{\tau}(n-1)\lambda(n) + (0,0,0,0,1,-1,0)^{\tau}, \tag{6}$$

$$\lambda(N) = (0,0,0,0,1,-1,0)^{\tau}, \tag{7}$$

for $n = N, \ldots 2$, where M^{τ} denotes the transpose of M, with $M(n) = \frac{\partial F(P(n))}{\partial P}$

continued

$$u_1^*(n) = \frac{1}{B_1}[t_p(\lambda_5(n+1) + \lambda_6(n+1) - \lambda_5(n+2) \\ -\lambda_6(n+2)) + \lambda_7(n+1) - \lambda_7(n+2)]$$

$$u_2^*(n) = \frac{1}{B_2}[\lambda_1(n+1) + \lambda_2(n+1) - \lambda_1(n+2) \tag{8}$$

$$-\lambda_2(n+2)] \tag{9}$$

and similar formulas for $n = 0$.

Idea

Use derivative of the map from controls-to-states to form the sensitivity operator and equations. Use the sensitivity operator and the form of the objective functional to find the adjoint system.

Use the adjoint system to simplify the quotient below and obtain OC characterizations

$$0 \leq \lim_{\epsilon \longrightarrow 0^+} \frac{J(u^* + \epsilon l) - J(u^*)}{\epsilon}$$

Numerical Implementation
Involves an iterative method with a forward sweep of the circulation model followed by a backward sweep of the adjoint model with a control characterization update afterwards.

The iterative method starts with a guess for the control values and then the control is updated after each forward sweep and backward sweep. The forward sweep and backward sweep are repeated until the convergence of the iterates is achieved.

Pressure Profiles

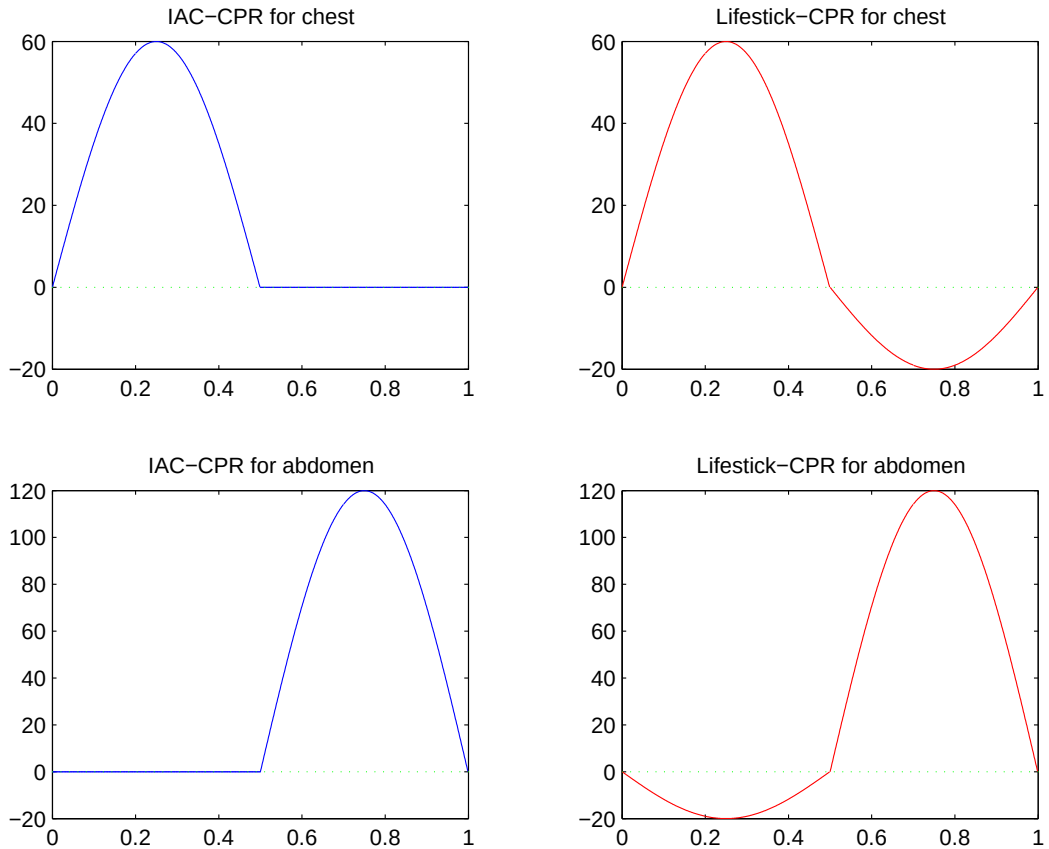


Figure: Each waveform represents one cycle.

Optimal Controls for Lifestick

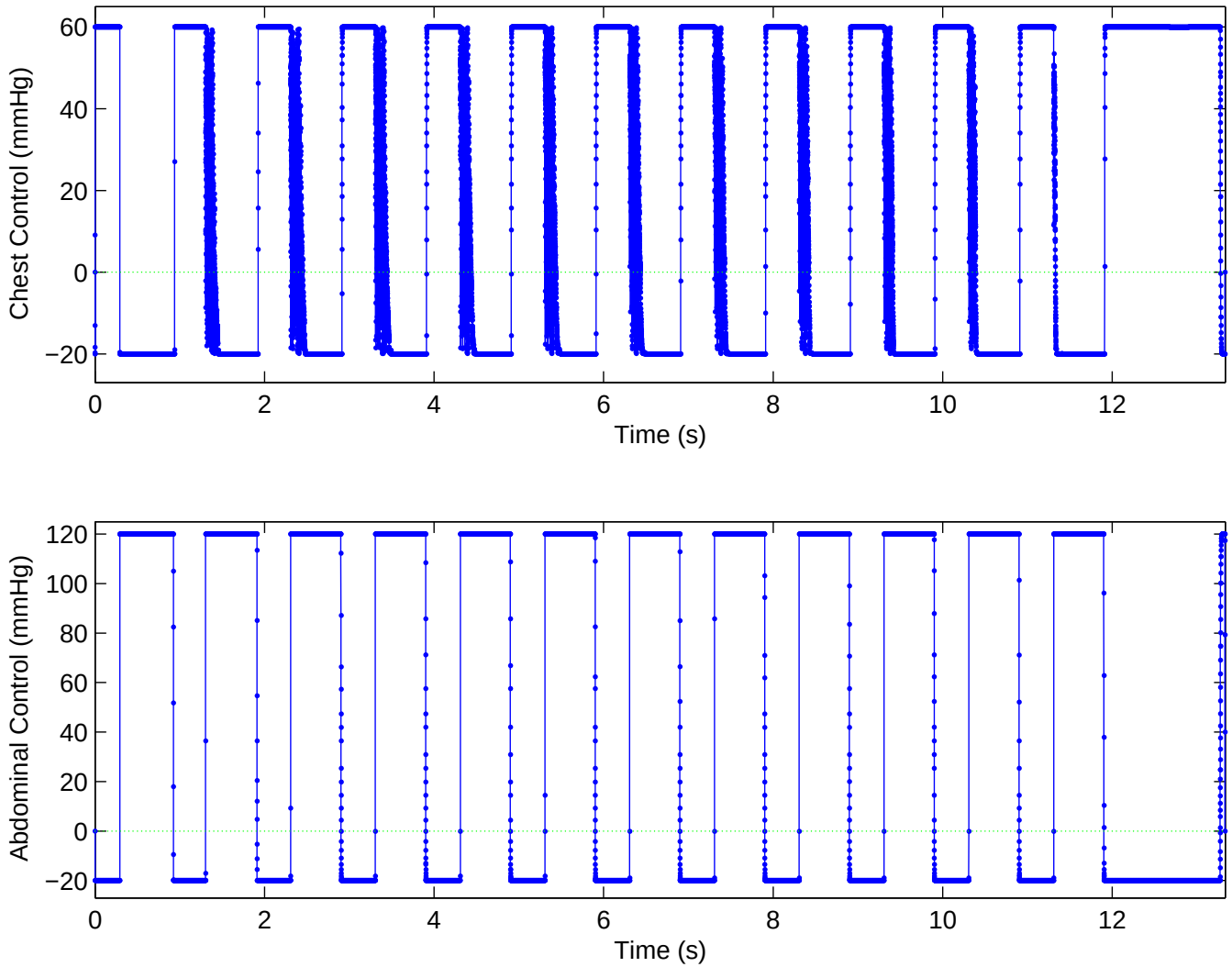


Figure: Controlled chest and abdominal pressure plotted top/bottom

Optimal Controls for IAC

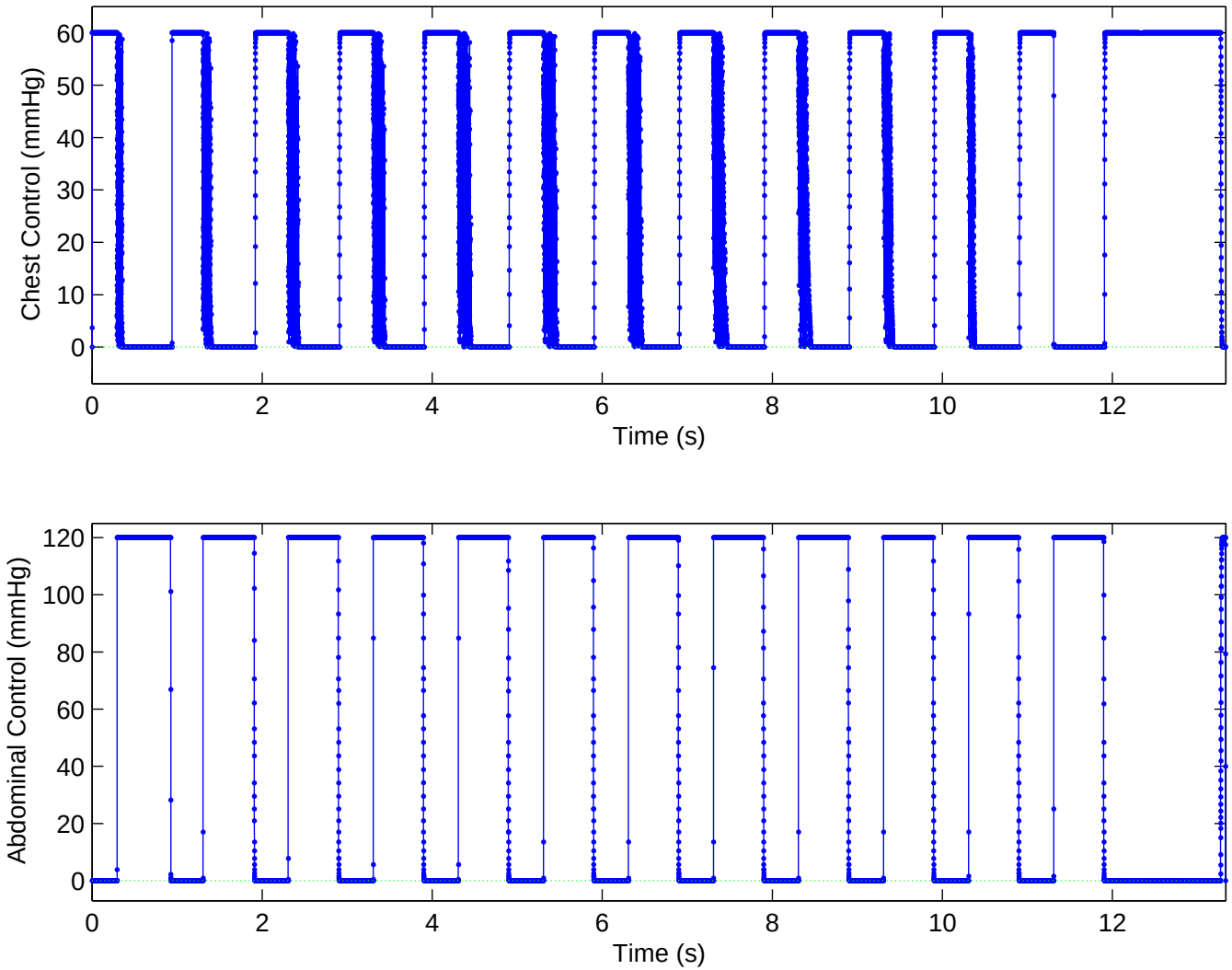


Figure: Controlled chest and abdominal pressure plotted top/bottom

Conclusion

Note that these results correspond with current suggested changes in CPR.

One can achieve about 20 to 30 percent improvement in the SPP.

Rabies in Raccoons

- Rabies is a common viral disease.
- Transmission through the bite of an infected animal.
- Raccoons are the primary vector for rabies in eastern US.
- Vaccine is distributed through food baits.

http://www.cdc.gov

Reported Cases of Raccoon Rabies, 2001

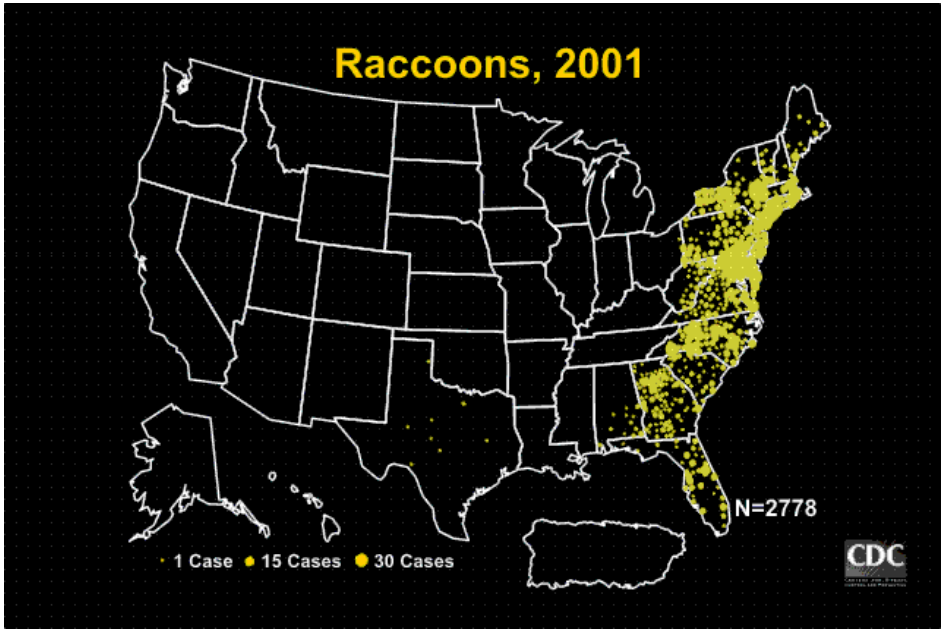


Figure: Reported Cases of Raccoon Rabies, 2001, *http://www.cdc.gov*

Costs and Treatment associated with Rabies in USA
30,000 persons/year given rabies post exposure prophylaxis at a cost of $30 million

Vaccination and prevention cost $300 million/year

In recent years, 8 million baits were distributed over 15 Eastern states.

Basic Assumptions

The objective of the problem formulation is to provide a simple, readily modified framework to analyze spatial optimal control for vaccine distribution as it impacts the spread of rabies among raccoons.

The epidemiological assumptions:

- No variance in time from infection to death
- Random mixing assumed to be the only means of contact and transmission

Temporal and Spatial Set-up

- Time scale: There is no population growth or immigration in the model presented here, but is included in a more general model. The scale is assumed to be over a time period (say within a season) over which births do not occur.
- Mortality occurs only due to infection.
- The time step of each iteration is that over which all infected raccoons die (e.g. about 10 days).
- Spatial scale: each cell is uniform in size, arranged rectangularly
- Movement: Raccoons are assumed to move according to a movement matrix from cell to cell, with distance dependence in dispersal.

Vaccine Vaccine/food packets are assumed to be reduced each time step due to uptake by raccoons, with the remaining packets then decaying due to other factors.

Then additional packets (CONTROL variable) are added at the end of each time step.

Variables

Model with (k,l) denoting spatial location, t time

- susceptibles = S(k,l,t)
- infecteds = I(k,l,t)
- immune = R(k,l,t)
- vaccine = v(k,l,t)
- control c(k, l, t), input of vaccine baits

Order of events
Within a time step (about a week to 10 days):

- First movement: using home range estimate to get range of movement. See sum S, sum I and sum R to reflect movement.
- Then: some susceptibles become immune by interacting with vaccine
- Lastly: new infecteds from the interaction of the non-immune susceptibles and infecteds

NOTE that infecteds from time step n die and do not appear in time step n + 1.

Susceptibles and Infecteds Equations

$$S(k,l,t+1) = (1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)$$
$$- \beta \frac{(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)},$$

Susceptibles and Infecteds Equations

$$S(k,l,t+1) = (1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t) - \beta \frac{(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)},$$

$$I(k,l,t+1) = \beta \frac{(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)}.$$

Immune and Vaccine Equations

$$R(k, l, t+1) = \text{sum\_}R(k, l, t) + e_1 \frac{v(k, l, t)}{v(k, l, t) + K} \text{sum\_}S(k, l, t),$$

$$v(k, l, t+1) =$$
$$Dv(k, l, t) \max\left[0, (1 - e(\text{sum\_}S(k, l, t) + \text{sum\_}R(k, l, t)))\right] + c(k, l, t).$$

States and Control

- States: S(m, n, t), I(m, n, t), R(m, n, t), v(m, n, t)
  for t = 2, ... T
  (given initial distribution at t = 1)
- Control c(m, n, t) , t= 1, 2, ..., T-1

Susceptibles and Infecteds Equations

$$S(k,l,t+1) = (1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)$$
$$- \beta \frac{(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)},$$

Susceptibles and Infecteds Equations

$$S(k,l,t+1) = (1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t) - \beta \frac{(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)},$$

$$I(k,l,t+1) = \beta \frac{(1 - e_1 \frac{v(k,l,t)}{v(k,l,t)+K})\text{sum\_}S(k,l,t)\text{sum\_}I(k,l,t)}{\text{sum\_}S(k,l,t) + \text{sum\_}R(k,l,t) + \text{sum\_}I(k,l,t)}.$$

Immune and Vaccine Equations

$$R(k, l, t+1) = \text{sum\_}R(k, l, t) + e_1 \frac{v(k, l, t)}{v(k, l, t) + K} \text{sum\_}S(k, l, t),$$

$$v(k, l, t+1) =$$
$$Dv(k, l, t) \max\left[0, (1 - e(\text{sum\_}S(k, l, t) + \text{sum\_}R(k, l, t)))\right] + c(k, l, t).$$

Objective Functional

maximize the susceptible raccoons, minimize the infecteds and cost of distributing baits

$$\sum_{m,n} \Big(I(m,n,T) - S(m,n,T) \Big) + B \sum_{m,n,t} c(m,n,t)^2,$$

where T is the final time and $c(m,n,t)$ is the cost of distributing the packets at cell (m,n) and time t, B is the balancing coefficient, c is the control.

Use discrete version of Pontryagin's Maximum Principle.

Hamiltonian at time t

$$
\begin{aligned}
H(m,n,t) = & B \sum_{m,n} c(m,n,t)^2 \\
& + \sum_{m,n} \Big[LS(m,n,t+1) \Big(\text{RHS of } S(m,n,t+1) \text{ eqn} \Big) \\
& + LI(m,n,t+1) \Big(\text{RHS of } I(m,n,t+1) \text{ eqn} \Big) \\
& + LR(m,n,t+1) \Big(\text{RHS of } R(m,n,t+1) \text{ eqn} \Big) \\
& + Lv(m,n,t+1) \Big(\text{RHS of } v(m,n,t+1) \text{ eqn} \Big) \Big]
\end{aligned}
$$

Adjoints and Optimal Control

LS, LI, LR, Lv denote the adjoints for S, I, R, v respectively

$$LS(i,j,t) = \frac{\partial H(t)}{\partial S(i,j,t)},$$

$$\frac{\partial H(t)}{\partial c(i,j,t)} = 2Bc(i,j,t) + Lv(i,j,t+1) = 0.$$

$$\Longrightarrow c^*(i,j,t) = -\frac{1}{2B} Lv(i,j,t+1),$$

subject to the upper and lower bounds

Numerical Iterative Method

- Start with a control guess and initial distribution of raccoons
- Solve the state equations forward
- Solve the adjoint equations backwards, using LI(k, I, T)=1, LS(k,I, T) =-1, other adjoints are zero at final time
- Update the control using the characterization
- Repeat until convergence

Disease Starts From the Corner: Initial Distribution

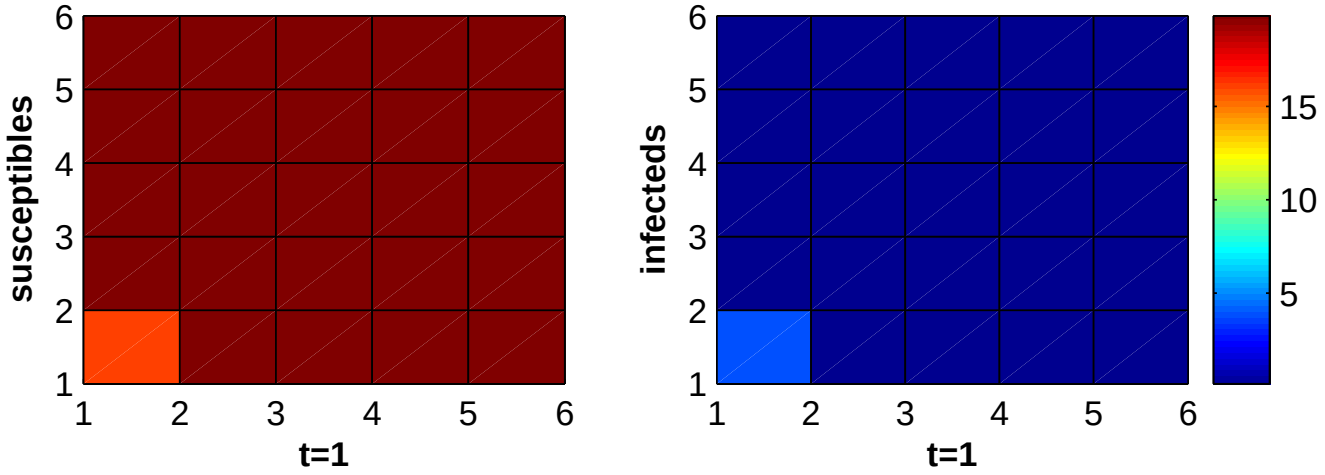


Figure: Initial Distribution

Susceptibles, no control

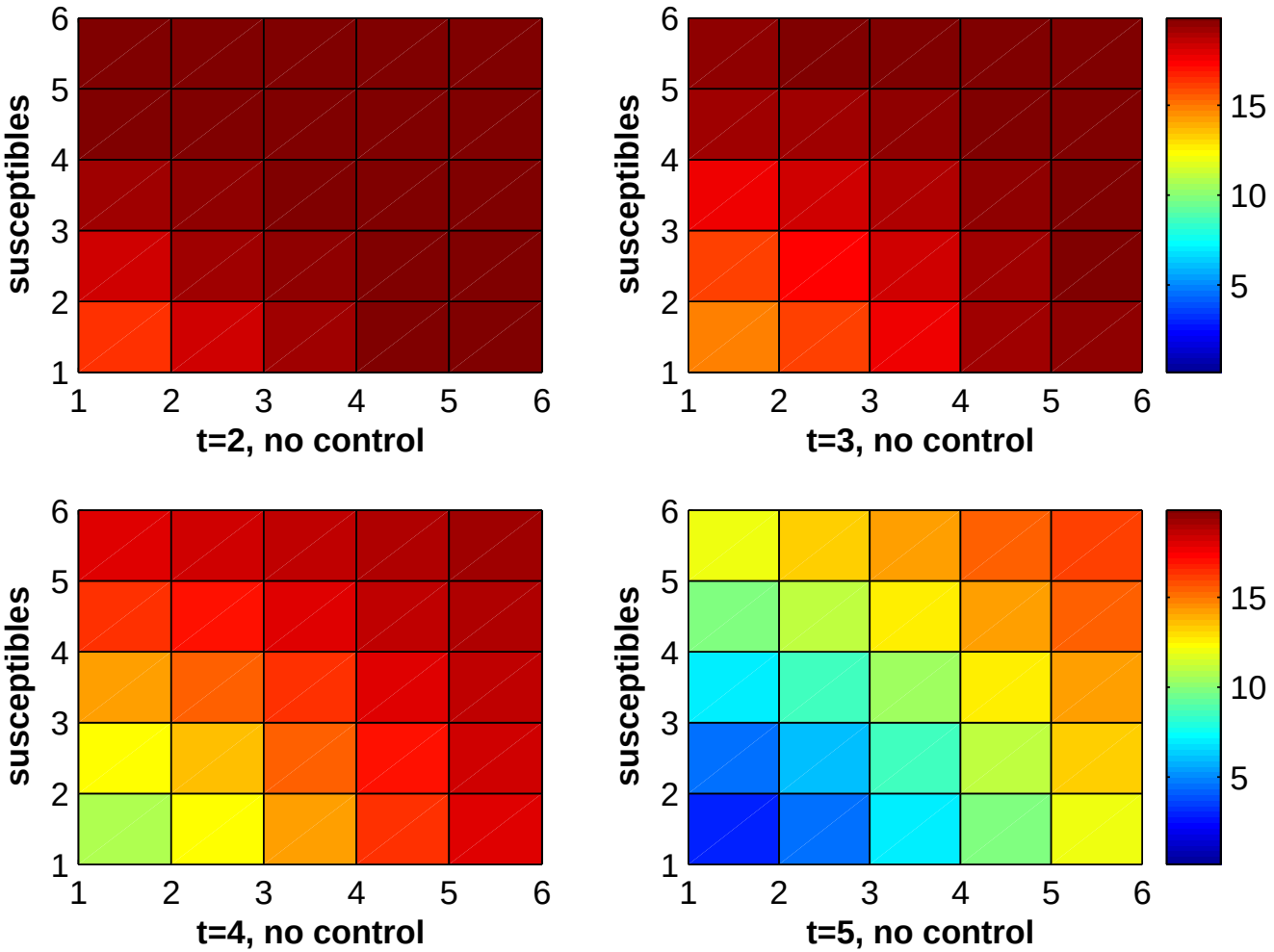


Figure: Susceptible Raccoons Without Control

Infecteds, no control

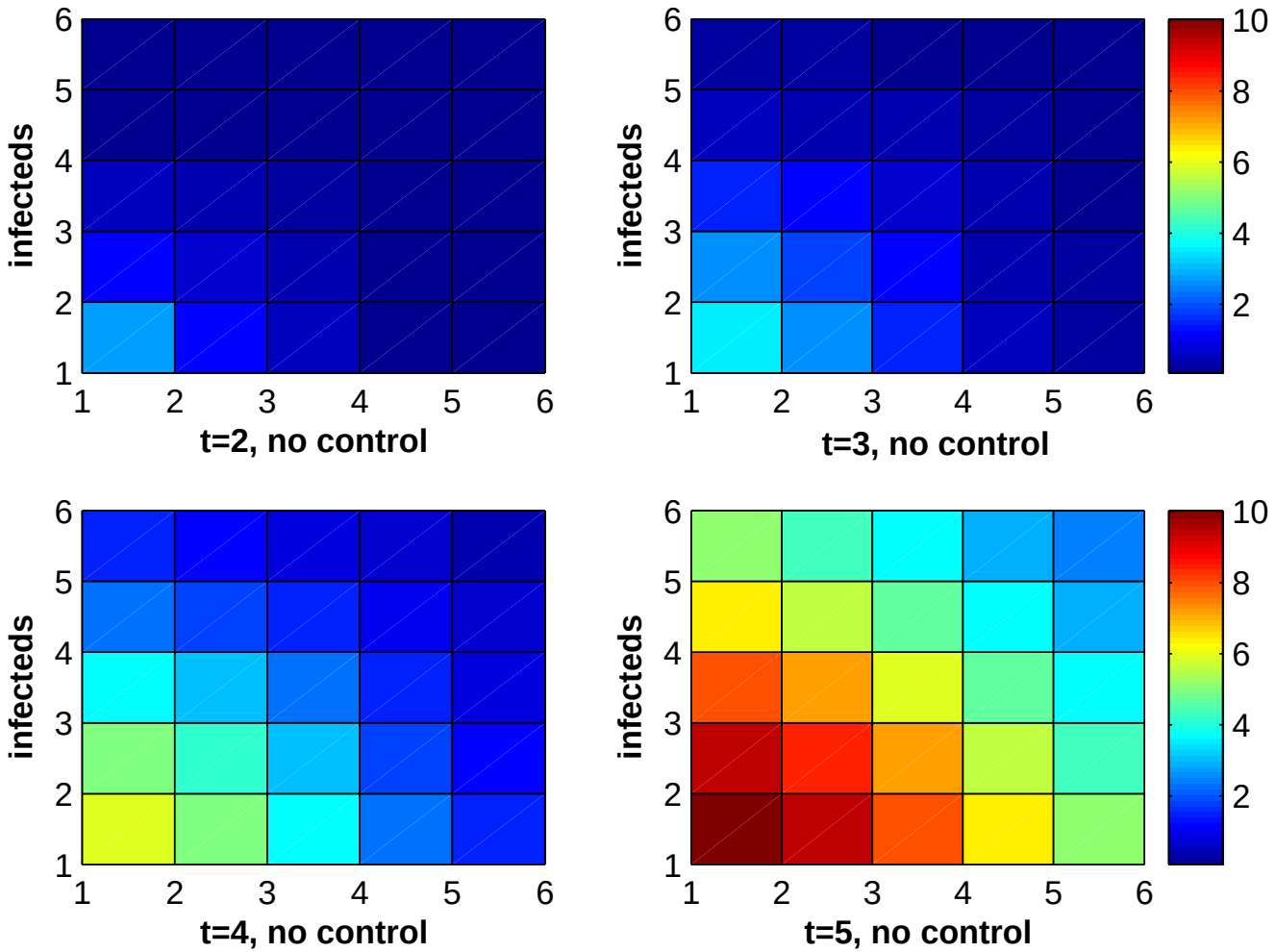


Figure: Infected Raccoons Without Control

Susceptibles, with control, $B = 0.5$

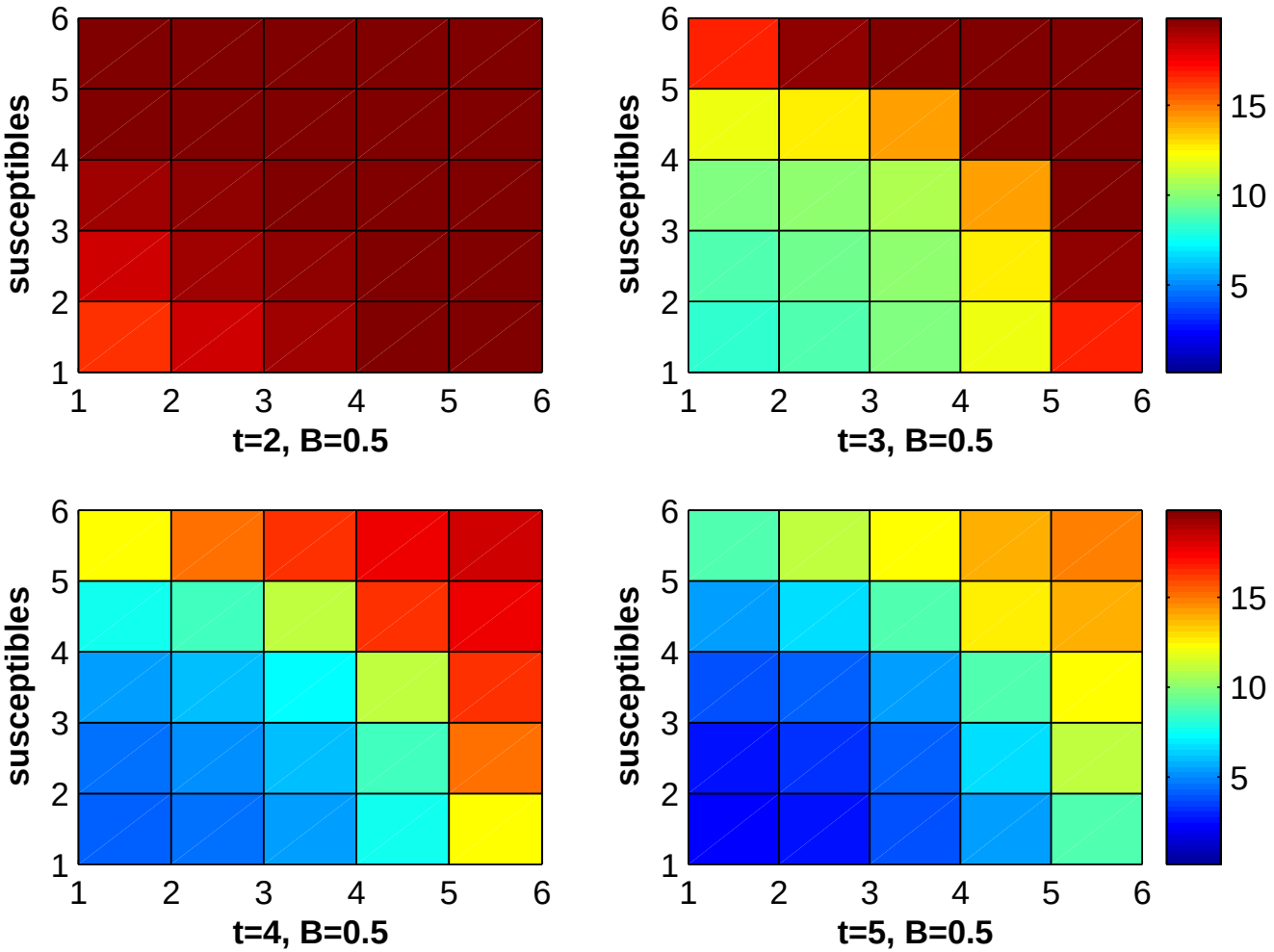


Figure: Susceptible Raccoons, $B = 0.5$

Infecteds, with control, $B = 0.5$

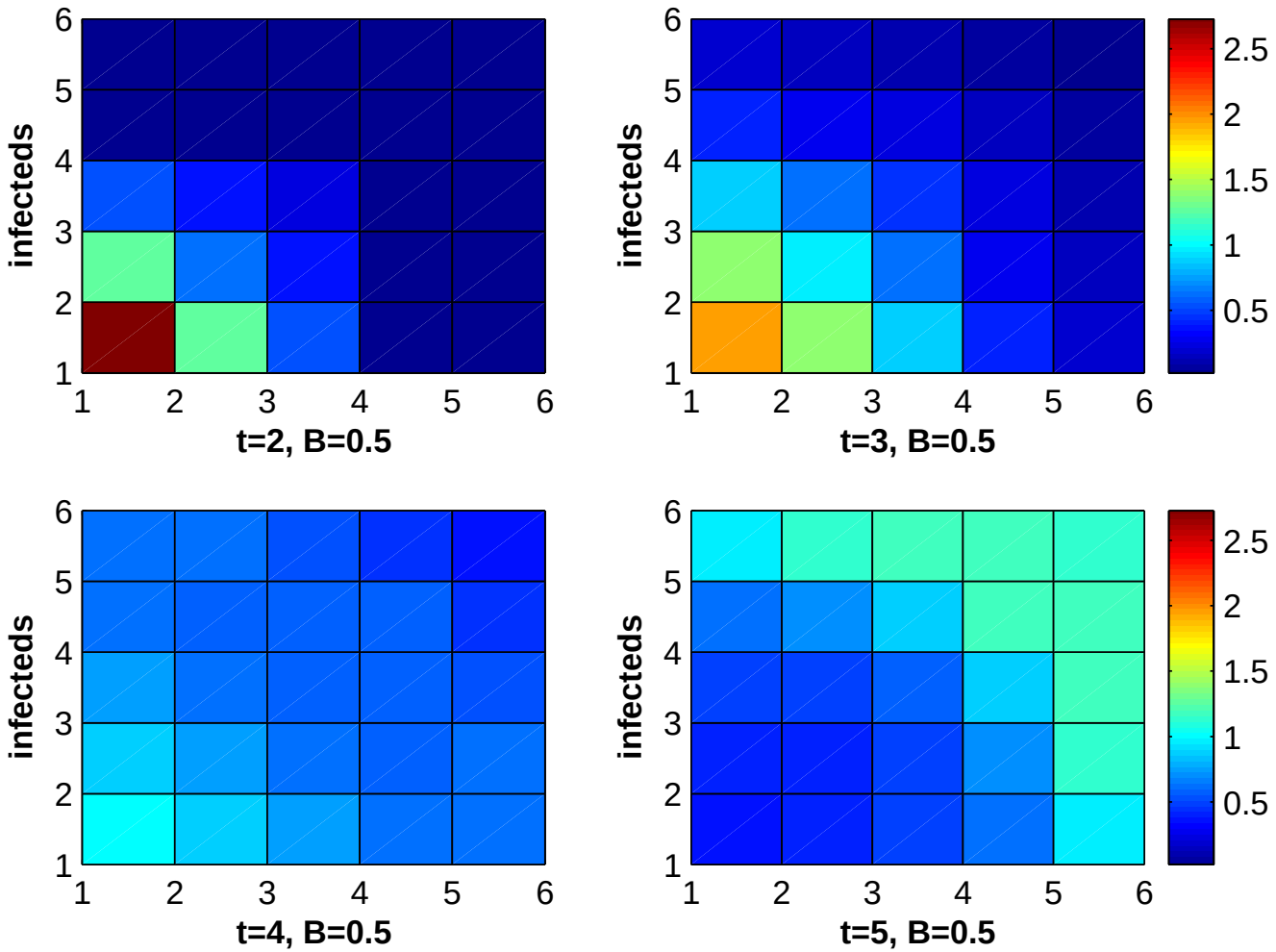


Figure: Infected Raccoons, $B = 0.5$

Immune, with control, $B = 0.5$

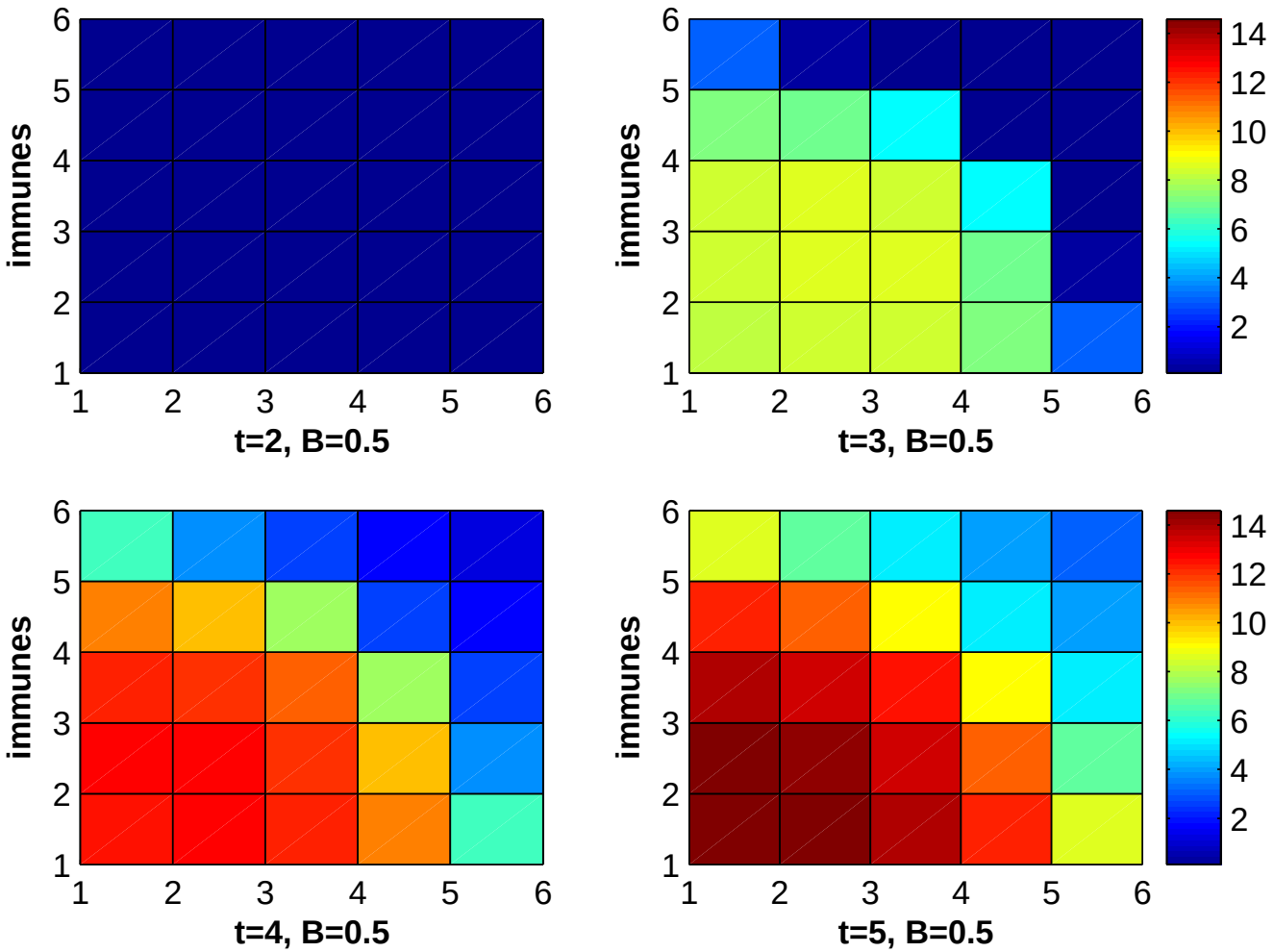


Figure: Immune Raccoons, $B = 0.5$

Optimal Control, $B = 0.5$

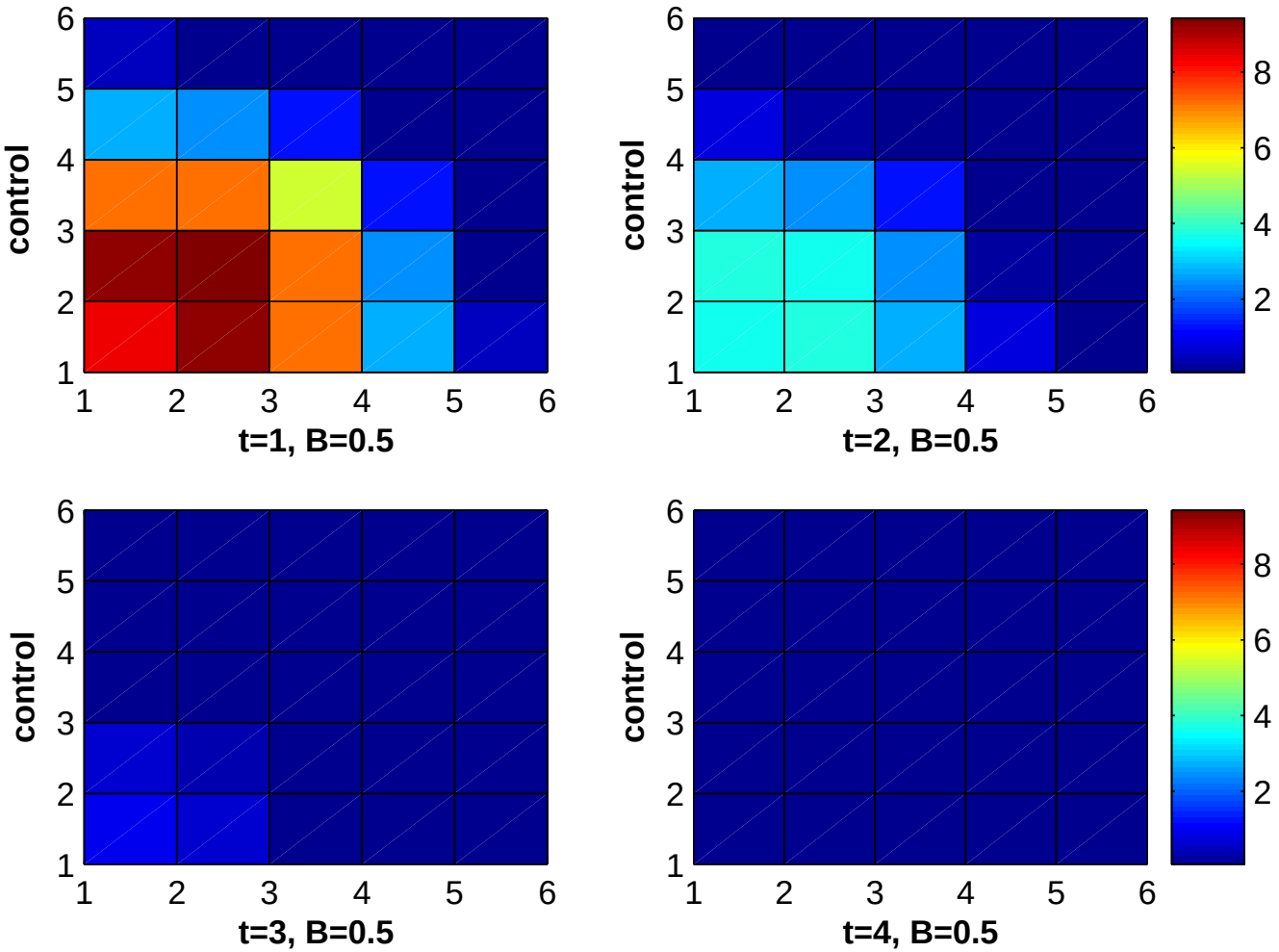


Figure: Optimal Control, $B = 0.5$

Disease Starts From the Center: Initial Distribution

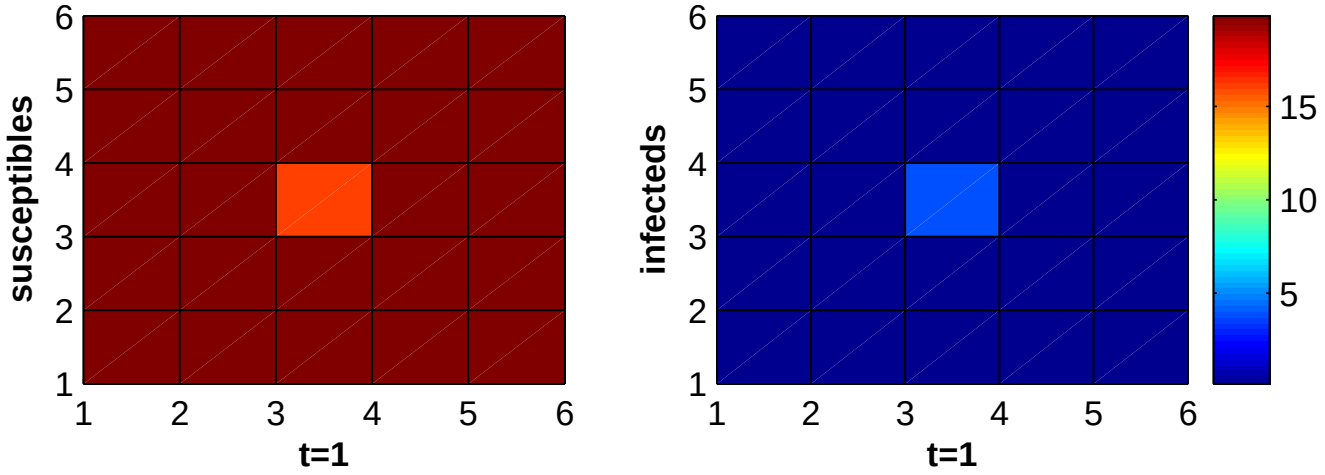


Figure: Initial Distribution

Susceptibles, $B = 0.5$

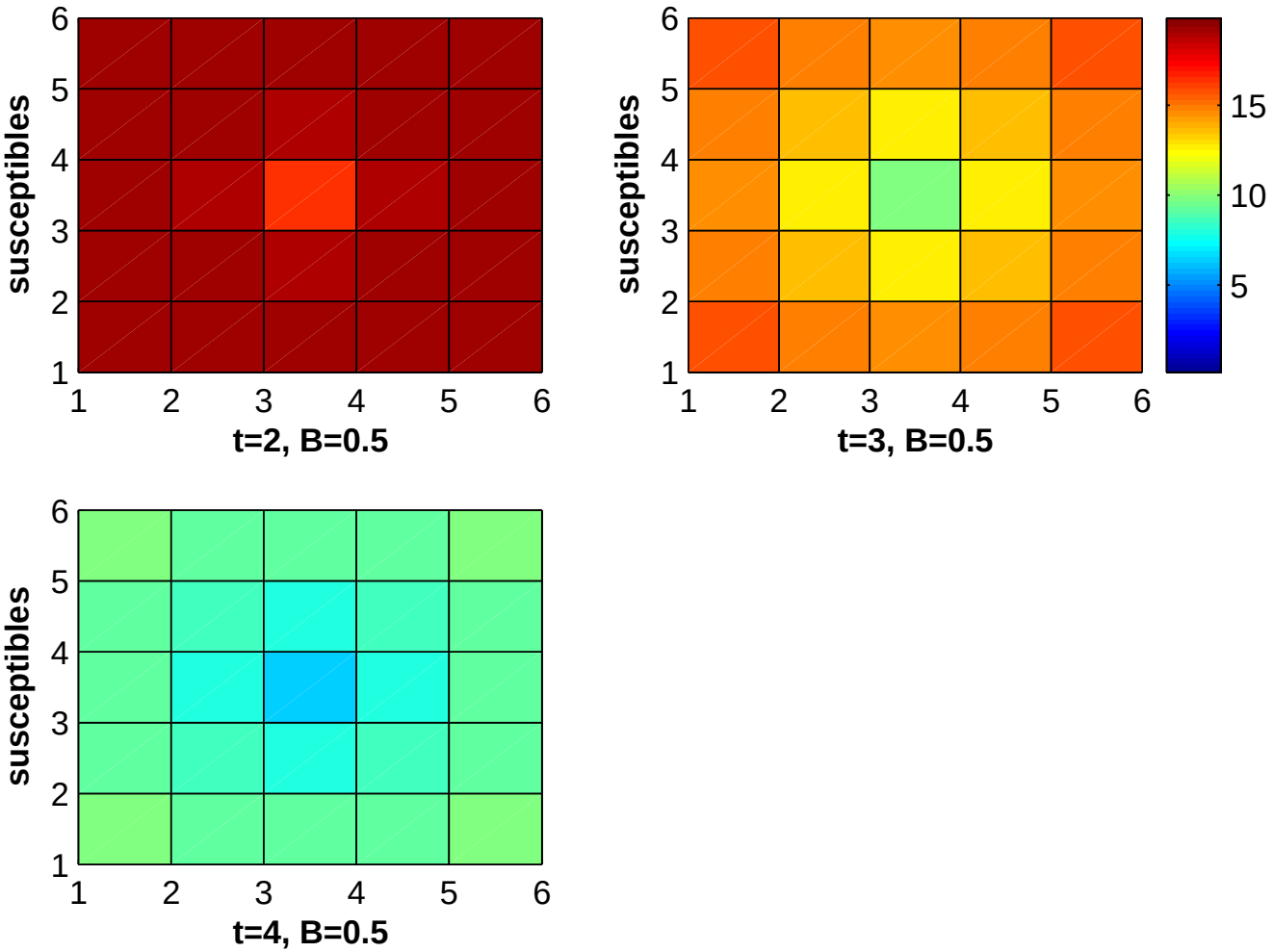


Figure: Susceptible Raccoons, $B = 0.5$

Infecteds, $B = 0.5$

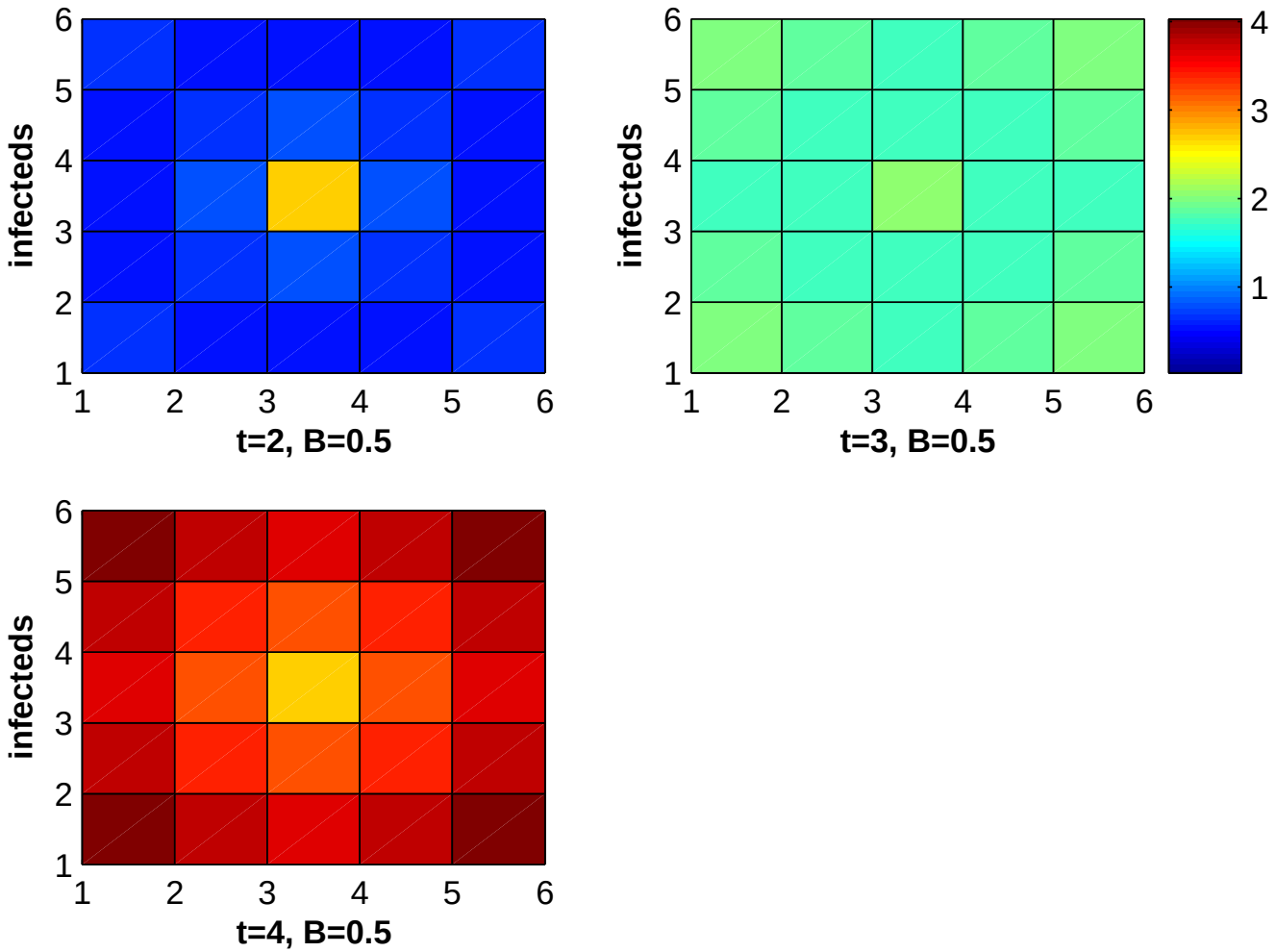


Figure: Infected Raccoons, $B = 0.5$

Optimal Control, $B = 0.5$, $t = 1$

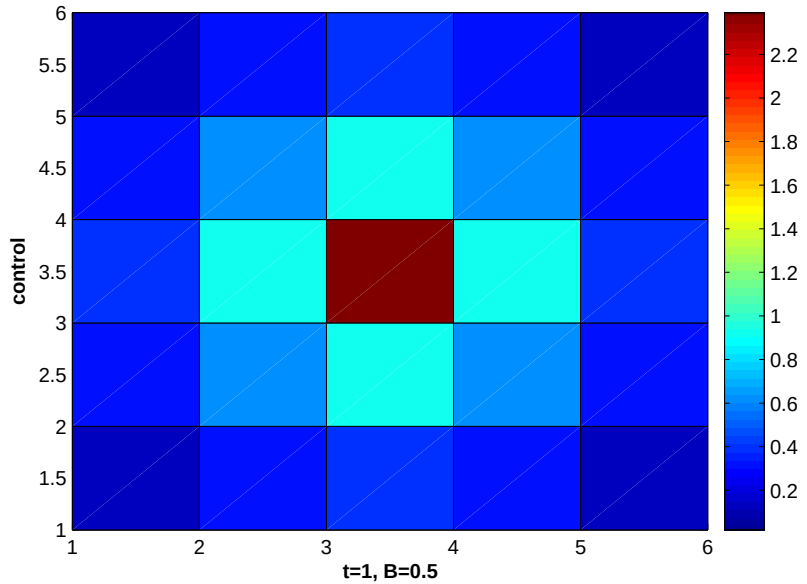


Figure: Optimal Control, $B = 0.5$, $t = 1$

Inhomogeneous Initial Distribution

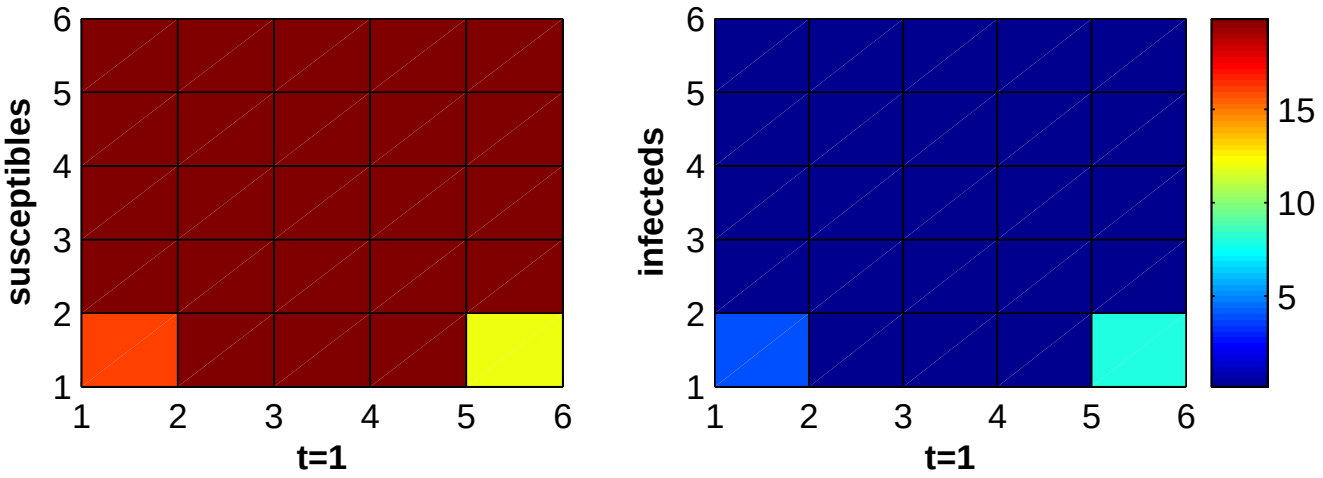


Figure: Inhomogeneous Initial Distribution

Optimal Control, $B = 0.5$, $t = 1$

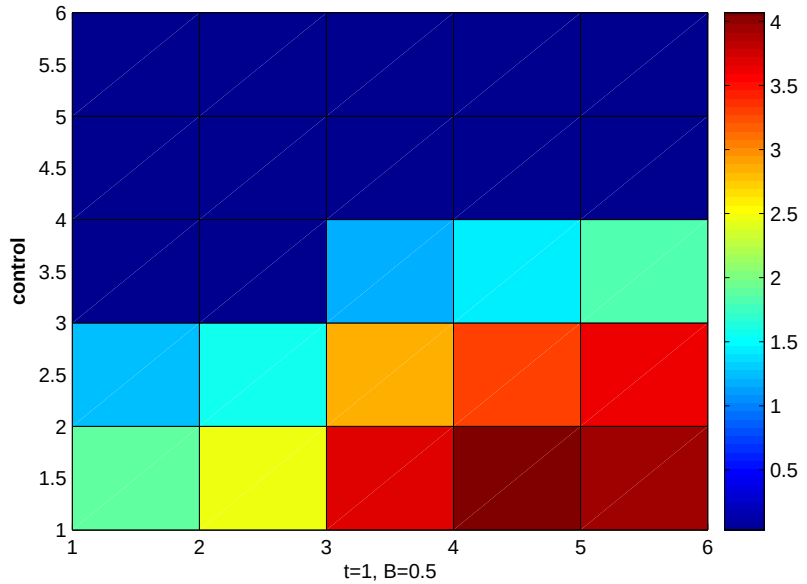
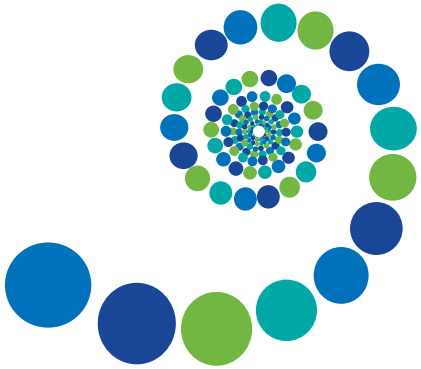


Figure: Optimal Control, $B = 0.5$, $t = 1$

Conclusion

- Developed a method and model to determine different optimal distributions of vaccine to control rabies spread;
- Illustrated the approach using three scenarios;
- Optimal bait distribution depends on the initial location of the disease outbreak and the distribution of raccoons throughout the grid;
- The method can be readily extended to evaluate optimal vaccination distribution strategies with other spatially heterogeneous interactions, larger spatial grids and different movement assumptions (including density dependence).
- Linear objective functional and see the resulting changes.



NIMBioS

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