Uncoupling Accuracy and Stablity in Tikhonov Regularization Parameter Selection

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Motivation

Ideal Problem: Solve $Ax = \bar{y} \in \text{Range}(A)$ for $x = x_{true}$

Actual Problem: Given $y = \bar{y} +$ NOISE

$$\text{Solve } Ax = y$$

- X, Y - Hilbert spaces; $A : X \to Y$ compact, linear operator
- compact $\Rightarrow A^{-1}$ is "unbounded"

Shaw problem

Example - Fredholm Integeral Equation of the 1st Kind
(ref. Shaw, Hansen)

$$\int_{-\pi/2}^{\pi/2} \kappa(t,\tau)x(\tau)dt = \bar{y}(\tau), \quad -\pi/2 \leq \tau \leq \pi/2,$$

$$\kappa(t,\tau) = (cos(t) + cos(\tau))^2 \left(\frac{sin(\lambda)}{\lambda}\right)^2,$$

$$\lambda = \pi(sin(t) + sin(\tau))$$

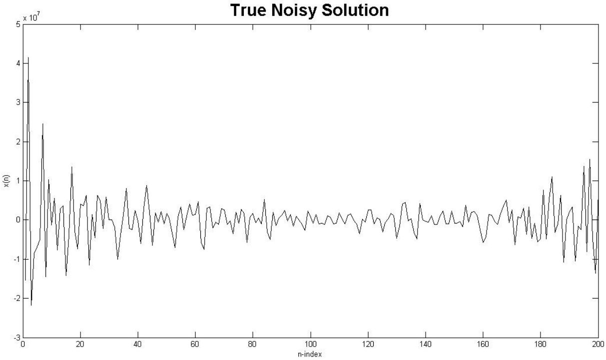
Problem Specifications

- $\bar{y} \in \mathbb{R}^{200}$, and x_{true} - sum of 2 Gaussian functions
- Condition number, $\kappa(A) \approx 5 \times 10^{19}$

Illustrative example

Noise $\approx 10^{-10}\%$

Approximate $x_{true} := A^{-1}\bar{y}$ by $x_{noise} := A^{-1}(\bar{y} + \text{ NOISE })$



Noise completely contaminates solution!

Background:

Tikhonov-Lavrentiev regularization:
Fix $0 < \alpha < 1$ (for stabilty AND accuracy),

$$\text{Solve for } x_0: \qquad (A + \alpha I)x_0 = y$$

Motivation for research efforts:

- (1948, 1963) A.N. Tikhonov, *Tikhonov Regularization*
- (1979) J.T. King, D. Chillingworth,
  Iterated Tikhonov (model formulation, error bounds)
- Current applications: Parameter estimation, turbulence modeling, imaging, etc.

Iterated Tikhonov Regularization

Algorithm

(Iterated Tikhonov-Lavrentiev regularization:)

- *Fix* $\alpha > 0$ *(for stability)*
- *Fix* $J > 0$ *(recover accuracy)*
- *Solve for* x_0,

$$(A + \alpha I)x_0 = y$$

- *For* $j = 1, 2, \ldots, J$, *solve for* x_j

$$(A + \alpha I)(x_j - x_{j-1}) = y - Ax_{j-1}$$

Error estimation

Theorem

Fix $J > 0$. *Fix* $0 \leq \beta \leq J$. *Let* $\epsilon =$ *NOISE.*

- **Regular solutions:** x_{true} *in* $Range(A^J)$ *and* $||\epsilon||_X \leq \epsilon_0$,

$$||e_J||_X \leq \frac{(J+1)\epsilon_0}{\alpha} + C(J)\alpha^{J+1}$$

- **Less regular solution:** x_{true} *in* $Range(A^\beta)$ *and* $||A^{J-\beta}\epsilon||_X \leq \epsilon_0$

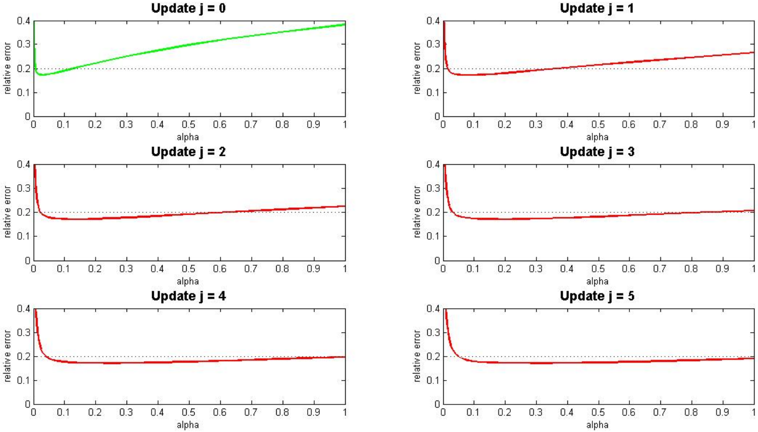
$$||e_J||_X \leq \frac{(\beta+1)\epsilon_0}{\alpha} + C(\beta)\alpha^{\beta+1}$$

(Recover optimal convergence in large-scales)

$$||A^{J-\beta}e_J||_X \leq \frac{(J+1)\epsilon_0}{\alpha} + C(J)\alpha^{J+1}$$

Illustrative example

Noise, $\epsilon \approx 10^{-1}\%$



Descent properties

Let $x_{noise} := A^{-1}(\bar{y} + \text{ NOISE})$, and $x_{true} = A^{-1}\bar{y}$

- x_{noise} minimizes $J_\epsilon(x) := \frac{1}{2}(Ax, x)_X - (\bar{y} + \text{ NOISE}, x)_X$
- x_{true} minimizes $J_0(x) := \frac{1}{2}(Ax, x)_X - (\bar{y}, x)_X$

Theorem

- x_j: **minimizing sequence of** $J_\epsilon(\cdot)$
- *Suppose* $||\epsilon||_X \leq \epsilon_0$ *and*

$$\alpha \geq \frac{\epsilon_0}{||x_{j+1} - x_j||_X}, \text{ (stopping criterion)}$$

x_j: **"minimizing sequence" of noise-free functional** $J_0(\cdot)$

Parameter Sensitivity

Definition

The sensitivity of the iterated Tikhonov updates is

$$s_j(\alpha) = \frac{dx_j(\alpha)}{d\alpha}$$

Theorem

- *Define* $T_k(x_0(\alpha)) := x_0(\alpha) - \alpha x_0'(\alpha) + \ldots + \frac{(-\alpha)^k}{k!} x_0^{(k)}(\alpha)$
- *For* $NOISE = 0$

$$x_{true} := x_0(\alpha = 0) = T_k(x_0(\alpha)) + O(\alpha^{k+1})$$

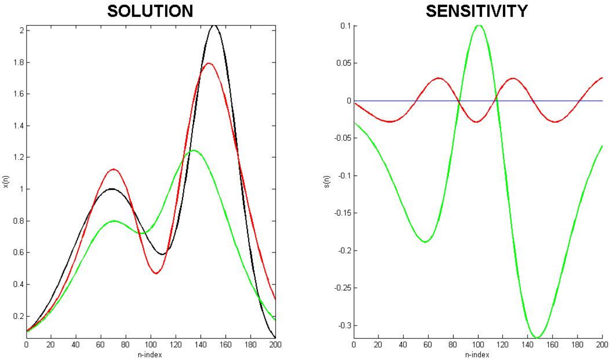
- *For* $NOISE \neq 0$

$$x_J(\alpha) = T_J(x_0(\alpha))$$

Numerical Results

$\alpha = 1.00, \; J = 80$

— $= x_{true}$, — $= x_0$, — $= x_J$

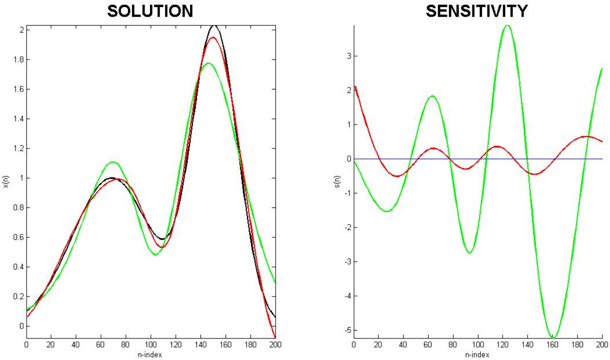


$s_0(min) \approx -0.3167, \quad s_0(max) \approx 0.1008$
$s_J(min) \approx -0.0289, \quad s_J(max) \approx 0.0301$

Numerical Results

$\alpha = 0.01$, $J = 80$

— $= x_{true}$, — $= x_0$, — $= x_J$

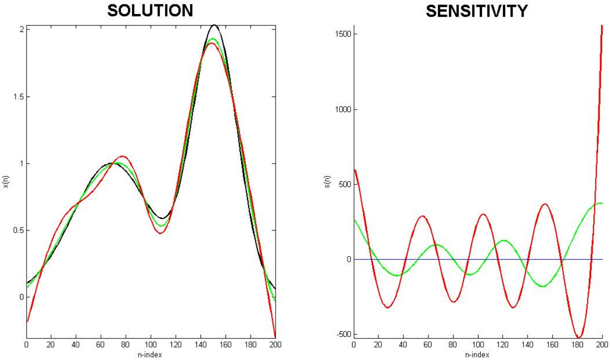


$s_0(min) \approx -5.2353$, $s_0(max) \approx 3.9044$
$s_J(min) \approx -0.5184$, $s_J(max) \approx 2.1215$

Numerical Results

$\alpha = 0.00001$, $J = 80$

— $= x_{true}$, — $= x_0$, — $= x_J$



$$s_0(min) \approx -1.822E+2, \quad s_0(max) \approx 3.725E+2$$
$$s_J(min) \approx -5.252E+2, \quad s_J(max) \approx 1.562E+3$$