Constructing a constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics, part 1

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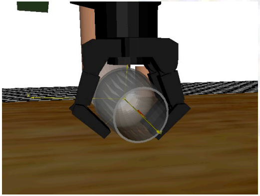
Application of Rigid Multi Body Dynamics

- RMBD in diverse areas
  - ⋆ rock dynamics
  - ⋆ robotic simulations
  - ⋆ virtual reality
  - ⋆ human motion
  - ⋆ nuclear reactors
  - ⋆ haptics
- VR or Virtual reality exposure (VRE) therapy
  - ⋆ fear of heights
  - ⋆ telerehabilitation
  - ⋆ fear of public speaking
  - ⋆ PTSD



Some Previous Approaches

- Integrate-detect-restart simulation a natural choice
  - Classical solution may not exist
  - Collisions can cause small stepsizes
- Differential algebraic equations (DAE) for joint constraints
  - Specialized techniques because non-smooth noninterpenetration and friction constraints.
- Optimization based animation technique solving a quadratic program at each step to avoid stiffness.
  - Collision detection still present, hence small stepsizes
- Penalty Barrier Methods are most popular.
  - Easy set up, even for DAEs, but problem may be stiff and requires *a priori* smoothing parameters

Hard Constraint Approaches

- Advantage:
  - Results are same order of magnitude as penalty method
  - Same dynamics using 4 orders of magnitude larger time step
  - We use a velocity impulse LCP based approach avoiding the lack of a solution and introducing artificial stiffness
- Disadvantage:
  - LCP model yields inequality constraints from contact and friction, treated computationally as hard constraints.

- To avoid infinitely small time steps, say from collisions, we need to impose a minimum stepsize

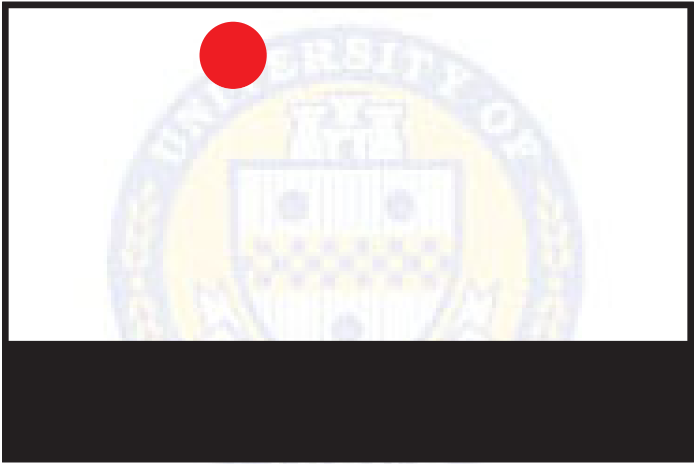
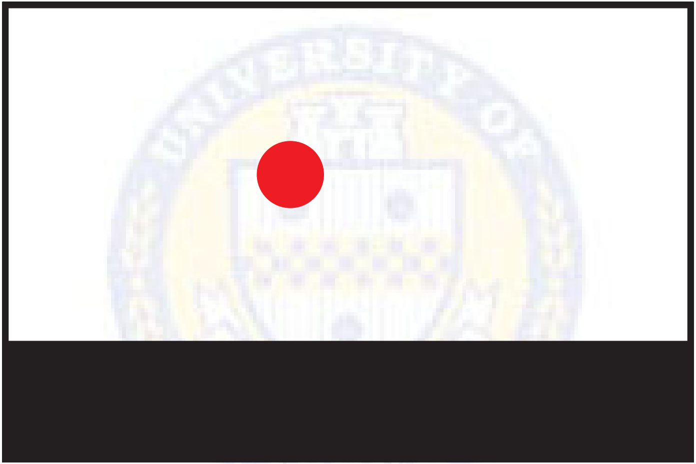


Figure: Simple Simulation: Trivial Example



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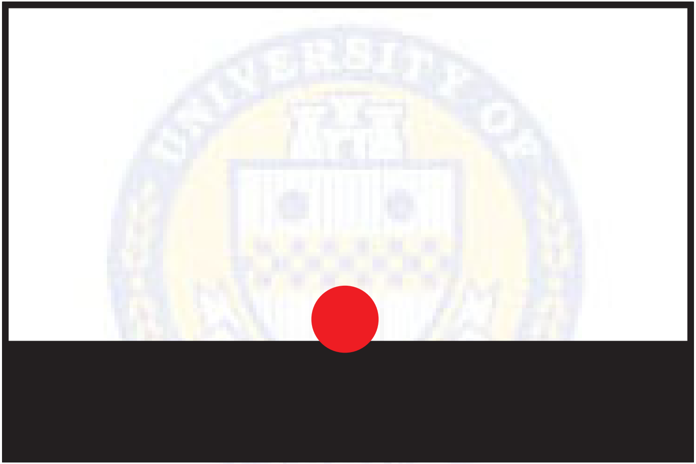

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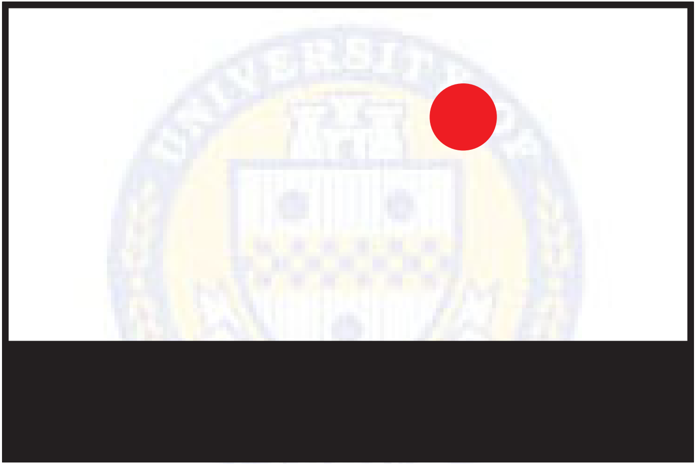
Previous Approaches



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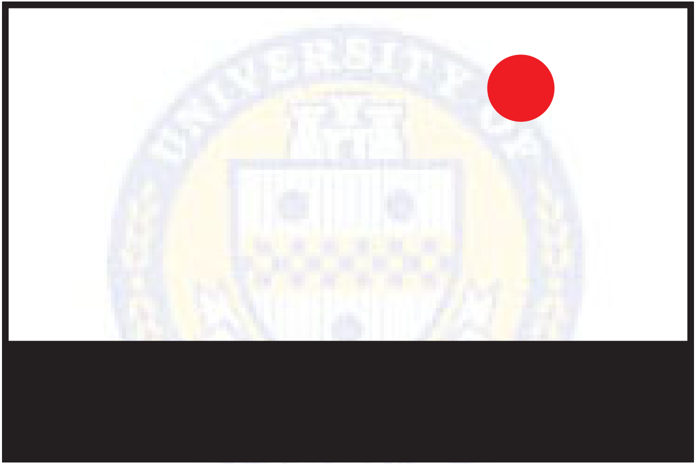

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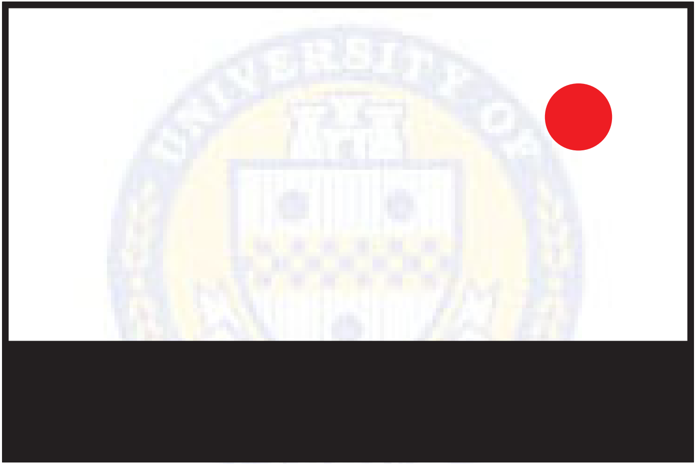

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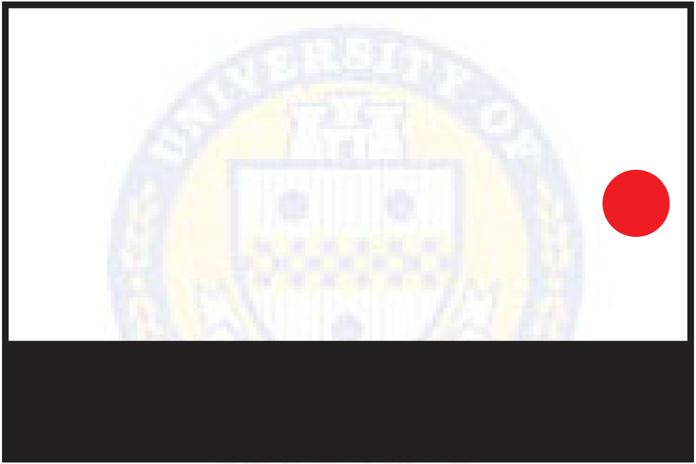

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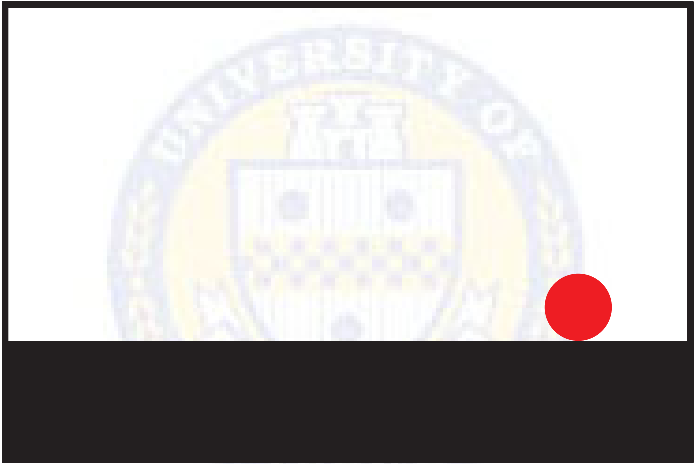

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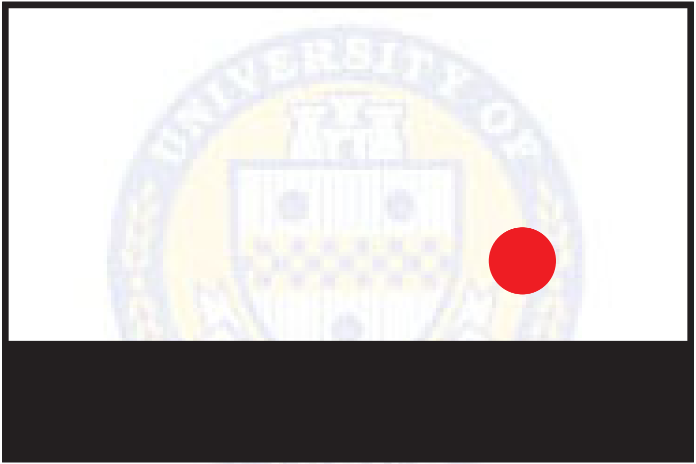

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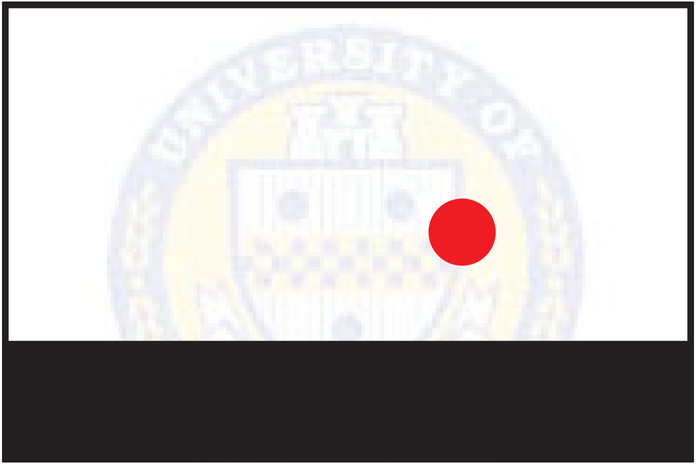


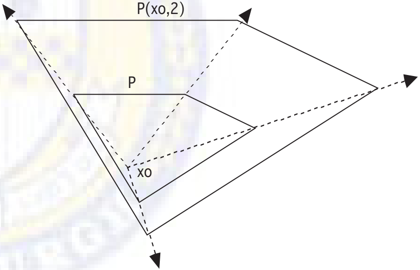
Figure: Simple Simulation: Trivial Example

Need to Define and Compute Depth of Penetration

- For methods with minimum time step, interpenetration may be unavoidable, thus it needs to be quantified (to limit amount of interpenetration)
- Minimum Euclidean distance good for distance between objects, but not for penetration

Construction of a constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

- Ratio Metric
- Differentiability
- Constraints and Model
- Algorithm
- Numerical Results
- Summary



Polyhedra and Expansion/Contraction Maps

Definition

We define CP(A, b, x_o) to be the convex polyhedron P defined by the linear inequalities $Ax \leq b$ with an interior point x_o. We will often just write P = CP(A, b, x_o).

Definition

Let P = CP(A, b, x_o). Then for any nonnegative real number t, the expansion (contraction) of P with respect to the point x_o is defined to be

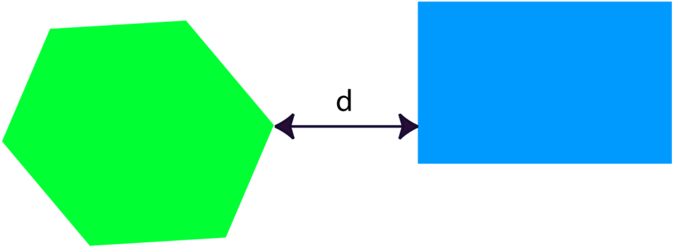
$$P(x_o, t) = \{x|Ax \leq tb + (1-t)Ax_o.\}$$

Minkowski Penetration Depth

Definition

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2. The Minkowski Penetration Depth (MPD) between the two bodies P_1 and P_2 is defined formally as

$$PD(P_1, P_2) = \min\{||d|| \,|\, interior(P_1 + d) \bigcap P_2 = \emptyset\}. \quad (1)$$



Ratio Metric Penetration Depth

Definition

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2. Then the Ratio Metric between the two sets is given by

$$r(P_1, P_2) = \min\{t | P_1(x_1, t) \bigcap P_2(x_2, t) \neq \emptyset\}, \tag{2}$$

and the corresponding Ratio Metric Penetration Depth (RPD) is given by

$$\rho(P_1, P_2, r) = \frac{r(P_1, P_2) - 1}{r(P_1, P_2)}. \tag{3}$$

Expansion/Contraction Again

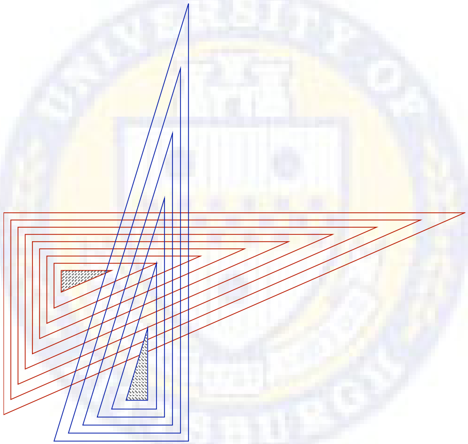


Figure: Visual representation of double expansion or contraction

Metric Equivalence Theorem

Theorem (Metric Equivalence)

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2, s be the MPD between the two bodies, D be the distance between x_1 and x_2, ϵ be the maximum allowable Minkowski penetration between any two bodies. Then the ratio metric penetration depth between the two sets satisfies the relationship

$$\frac{s}{D} \leq \rho(P_1, P_2, r) \leq \frac{s}{\epsilon}, \tag{4}$$

if P_1 and P_2 have disjoint interiors, and

$$-\frac{s}{\epsilon} \leq \rho(P_1, P_2, r) \leq -\frac{s}{D} \tag{5}$$

if the interiors of P_1 and P_2 are not disjoint.

Significance of the Metric Equivalence Theorem

- Let number of facets of two polyhedra be m_1 and m_2
  - Computing PD by using the Minkowski sums: $O(m_1^2 + m_2^2)$
  - Solving linear programming problem: $O(m_1 + m_2)$
- $\therefore$ our metric provide us with a simple way to detect collision and measure penetration of two convex polyhedral bodies bodies with lower complexity and is equivalent, for small penetration, to the classical measure
- $\therefore$ for time step h, if the MPD is $O(h^2)$ then so is the RPD

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$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(c\, u) = c\, \frac{du}{dx}$$

$$\frac{d}{dx}(u\, v) = u\, \frac{dv}{dx} + v\, \frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\, \frac{du}{dx} - u\, \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(u^n) = n u^{n-1} \frac{du}{dx}$$

$$\frac{d}{dx}(u \circ v) = \frac{dv}{dx}\left(\frac{du}{dx} \circ v\right)$$

Perfect Contact

Definition

Two convex polyhedra are in perfect contact when there is a nonempty intersection without interpenetration.

Definition

In n-dimensional space, a Basic Contact Unit (BCU) occurs when

- two convex polyhedra are in perfect contact,
- the contact region attached to a BCU is a point, and
- exactly n+1 facets are involved at the contact.

The point where the contact occurs is called an event point, or more simply, an event.

Basic Contact Unit

- A CoF is always a BCU
- In 2D: CoF In 3D: CoF, (nonparallel) EoE
- In n-dim space, there are exactly $\left[\frac{n+1}{2}\right]$ distinct BCUs

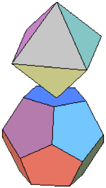


Figure: Corner-on-Face

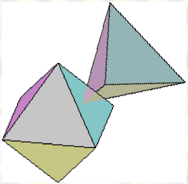


Figure: Edge-on-Edge

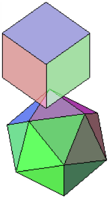


Figure: Face-on-Face

Convex Hull of BCUs

Theorem

The intersection of two convex polyhedra in perfect contact is the convex hull of the event points.

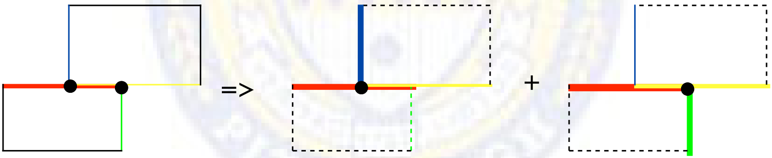
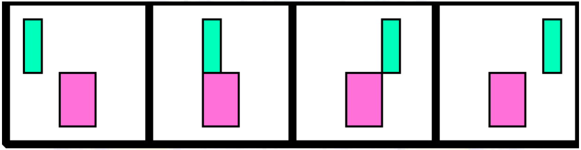


Figure: 2D Example: Contact Region Is Convex Hull of BCUs.

Nondifferentiability



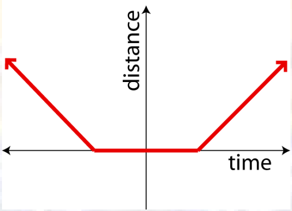


Figure: Nondifferentiability of Euclidean distance function

Infinite Differentiability at an Event

- If E is an event at perfect contact of convex polyhedra P_1 and P_2, then $P_E(x_i, t)$, the restrictions of $P_i(x_i, t)$ to E, is the convex body defined by the facets of $P(x_i, t)$ which involve E.

- If E is an event at perfect contact of P_1 and P_2, then

$$r(P_E(x_1, t), P_E(x_2, t)) = \min_{t \geq 0} \begin{cases} \hat{A}_{L_1} R_1^T x - \hat{b}_1 t \leq \hat{A}_{L_1} R_1^T x_1 \\ \hat{A}_{L_2} R_2^T x - \hat{b}_2 t \leq \hat{A}_{L_2} R_2^T x_2 \end{cases} \quad (6)$$

  where the sum of the rows of $\hat{A}_{L_1}$ and $\hat{A}_{L_2}$ totals n+1.

- Theorem: At any event E of perfect contact, $r(P_E(x_1, t), P_E(x_2, t))$ is infinitely differentiable with respect to the translation vectors and rotation angles.

Component Functions

- Associate m^{th} event $E^{(m)}$ with component function $\widehat{\Phi}^{(m)}$
- We use the restrictions $P_{E^{(m)}}(x_1, t)$ and $P_{E^{(m)}}(x_2, t)$
- $\widehat{\Phi}^{(m)} = f(r_m)$, where $f(t) = (t-1)/t$ and

$$r_m = \min_{t \geq 0} \left\{ \begin{array}{l} \hat{A}_{m_1} R_1^T x - b_{m_1} t \leq \hat{A}_{m_1} R_1^T x_1 \\ \hat{A}_{m_2} R_2^T x - b_{m_2} t \leq \hat{A}_{m_2} R_2^T x_2 \end{array} \right. \tag{7}$$

and sum of numbers of rows of $\hat{A}_{m_1}$ and $\hat{A}_{m_2}$ is n+1.

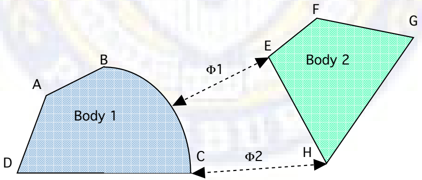


Figure: Two Component Signed Distance Functions

Max of Component Functions

RPD is the maximum of component distance functions.

Theorem

Suppose $x_1 \neq x_2$ *and let* $P_i = CP(A_{L_i}R_i^T, b_{L_i} + A_{L_i}R_i^T x_i, x_i)$ *be convex polyhedra for* $i = 1, 2$ *and let* $\left\{E^{(1)}, E^{(2)}, \cdots, E^{(N)}\right\}$ *be the list of all possible events with corresponding component distance functions* $\left\{\widehat{\Phi}^{(1)}, \widehat{\Phi}^{(2)}, \cdots, \widehat{\Phi}^{(N)}\right\}$. *Then*

$$\rho(P_1, P_2, r) = \max\left\{\widehat{\Phi}^{(1)}, \widehat{\Phi}^{(2)}, \cdots, \widehat{\Phi}^{(N)}\right\},$$

where $\rho(P_1, P_2, r)$ *is defined by (3).*

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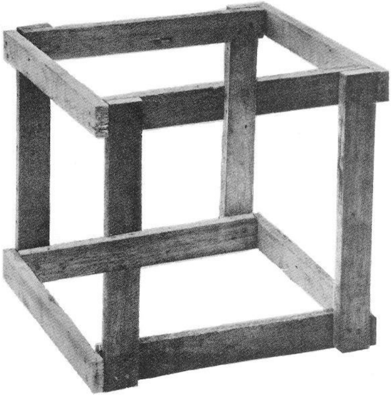

Noninterpenetration Constraints

- Model noninterpenetration constraints by continuous piecewise differentiable signed distance functions:

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \cdots, p. \tag{8}$$

- We will use RPD to compute $\Phi^{(j)}$

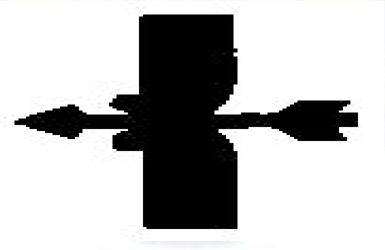

Figure: Noninterpenetration Constraint: Constraint not enforced

Joint Constraints

- Model joint constraints by sufficiently smooth $\Theta^{(i)}(q) = 0,\ i = 1, 2, \cdots, n_J$
- Define $\nu^{(i)}(q) = \nabla_q \Theta^{(i)}(q), \quad i = 1, 2, \cdots, n_J$

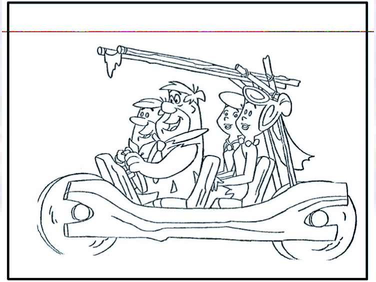

Figure: Joint Constraint: Fixed distance between wheels

Linear Complementarity Model

Euler discretization of the equations of motion:

$$\begin{aligned} M(q^{(l)})\left(v^{(l+1)} - v^{(l)}\right) &= h_l k\left(t^{(l)}, q^{(l)}, v^{(l)}\right) + \textstyle\sum_{i=1}^{n_J} c_\nu^{(i)} \nu^{(i)}(q^{(l)}) \\ &+ \sum_{m \in \mathcal{E}} \left(c_n^{(m)} n^{(m)}(q^{(l)}) + \sum_{i=1}^{M_C^{(m)}} \beta_i^{(m)} d_i^{(m)}(q^{(l)}) \right). \end{aligned} \tag{9}$$

Modified linearization of geometrical and noninterpenetration constraints:

$$\begin{array}{llll} \gamma \Theta^{(i)}(q^{(l)}) + h_l {\nu^{(i)}}^T(q^{(l)}) v^{(l+1)} & = 0, & i = 1, 2, \cdots, n_J, & \\ {n^{(m)}}^T(q^{(l)}) v^{(l+1)} + \frac{\gamma}{h_l} \Phi^{(j)}(q^{(l)}) & \geq 0 & \perp c_n^{(m)} \geq 0, & m \in \mathcal{E}. \end{array} \tag{10}$$

Friction Model

Friction model (usual classical pyramid approximation of friction cone, see Stewart & Trinkle 1995 or Anitescu & Hart 2004):

$$\begin{aligned} D^{(m)^T}(q)v + \lambda^{(m)} e^{(m)} &\geq 0 \quad \perp \quad \beta^{(m)} \geq 0, \\ \mu c_n^{(m)} - e^{(m)^T} \beta^{(m)} &\geq 0 \quad \perp \quad \lambda^{(m)} \geq 0. \end{aligned} \tag{11}$$

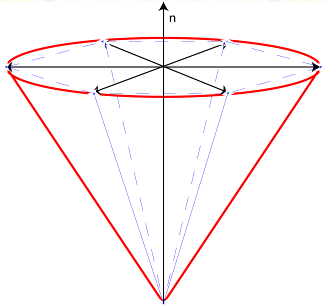

Figure: Approximation of Friction Cone

Mixed Complementarity and QP Formulation

$$\begin{array}{lllllll}
M^{(l)}v & -\widetilde{n}\widetilde{c}_n & -\widetilde{D}\widetilde{\beta} & & = -q^{(l)} & & \\
\widetilde{\nu}^T v & & & & = -\Upsilon & & \\
\widetilde{n}^T v & & & -\tilde{\mu}\lambda & \geq -\Gamma - \Delta & \perp & c_n \geq 0 \\
\widetilde{D}^T v & & & +\widetilde{E}\lambda & \geq 0 & \perp & \widetilde{\beta} \geq 0 \\
 & \tilde{\mu}c_n & -\widetilde{E}^T\widetilde{\beta} & & \geq 0 & \perp & \lambda \geq 0
\end{array} \qquad (12)$$

Note (12) constitutes 1^{st}-order optimality conditions of QP

$$\begin{array}{llll}
\min\limits_{v,\lambda} & \frac{1}{2}v^T M^{(l)} v + q^{(l)^T} v & & \\
\text{s.t.} & n^{(m)^T} v - \mu^{(m)}\lambda^{(m)} \geq -\Gamma^{(m)} - \Delta^{(m)}, & m \in \mathcal{E} & \\
 & D^{(m)^T} v + \lambda^{(m)} e^{(m)} \geq 0, & m \in \mathcal{E} & \\
 & \nu_i^T v = -\Upsilon_i, & 1 \leq i \leq n_J & \\
 & \lambda^{(m)} \geq 0 & m \in \mathcal{E} &
\end{array} \qquad (13)$$

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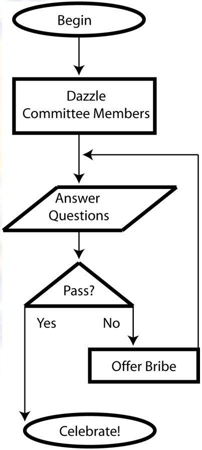

Assumption A1

A1: There exists $\epsilon_o > 0$, $C_1^d > 0$, and $C_2^d > 0$ such that

- $\Phi^{(j)}$ for $1 \leq j \leq n_B$ are piecewise continuous on their domains Ω_ϵ, with piecewise components $\widehat{\Phi}^{(m)}(q)$ which are twice continuously differentiable in their respective open domains with first and second derivatives uniformly bounded by $C_1^d > 0$ and $C_2^d > 0$, respectively, and

- $\Theta^{(i)}(q)$ for $i = 1, 2, \cdots, m$ are twice continuously differentiable in Ω_ϵ with first and second derivatives uniformly bounded by $C_1^d > 0$ and $C_2^d > 0$, respectively.

Using Assumption A1

Lemma

If Assumption A1 holds, then $\Phi^{(j)}$ for $1 \leq j \leq n_B$ is everywhere directionally differentiable. Moreover, the generalized gradient of $\Phi^{(j)}$ is contained in the convex cover of the gradients of its component functions which are active at q evaluated at q.

Note: We use $\Phi^{(j)^o}(q; v) = \limsup_{p \to q, t \downarrow 0} \dfrac{\Phi^{(j)}(p + tv) - \Phi^{(j)}(p)}{t}$

Lemma

If Assumption A1 holds, then for any j such that $1 \leq j \leq n_B$, then $\Phi^{(j)}$ satisfies a Lipschitz condition.

Note: We use Lebourg's Mean Value Theorem in the proof

Assumptions D1 - D3

D1: The mass matrix is constant. That is, $M\left(q^{(l)}\right) = M^{(l)} = M$.

D2: The norm growth parameter is constant: $c(\cdot,\cdot,\cdot) \leq c_o$

D3: The external force is continuous and increases at most linearly with the pos. and vel., and unif. bdd in time:

$$k(t,v,q) = k_o(t,v,q) + f_c(v,q) + k_1(v) + k_2(q)$$

and there is some constant $c_K \geq 0$ such that

$$\begin{aligned} \|k_o(t,v,q)\| &\leq c_K \\ \|k_1(v)\| &\leq c_K \|v\| \\ \|k_2(q)\| &\leq c_K \|q\| . \end{aligned}$$

Also assume

$$v^T f_c(v,q) = 0 \quad \forall v, q.$$

Algorithm for Piecewise Smooth RMBD

Algorithm

Algorithm for piecewise smooth multibody dynamics

Step 1: Given $q^{(l)}$. $v^{(l)}$. and h_l, calculate the active set $\mathcal{A}\left(q^{(l)}\right)$ and active events $\mathcal{E}\left(q^{(l)}\right)$.

Step 2: Compute $v^{(l+1)}$, the velocity solution of our mixed LCP .

Step 3: Compute $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$.

Step 4: IF finished, THEN stop ELSE set $l = l + 1$ and restart.

Main Result

Theorem

Assume that our algorithm is applied over a time interval [0, T], and

- *The active set* $\mathcal{A}(q)$ *and active events* $\mathcal{E}(q)$ *are properly defined*
- *The time steps* $h_l > 0$ *satisfy*
$$\sum_{l=0}^{N-1} h_l = T \quad \text{and} \quad \frac{h_{l-1}}{h_l} = c_h, \quad l = 1, 2, \cdots, N-1$$
- *The system satisfies Assumptions (A1) and (D1) - (D3)*
- *The system is initially feasible. That is,* $I(q^{(0)}) = 0$

Then, there exist $H > 0$, $V > 0$, *and* $C_c > 0$ *such that*
$$\left|\left|v^{(l)}\right|\right| \leq V \quad \text{and} \quad I\left(q(l)\right) \leq C_c \left|\left|v^{(l)}\right|\right|^2 h_{l-1}^2, \ \forall l, \ 1 \leq l \leq N$$

Consequences of the Theorem

- Algorithm achieves constraint stabilization because the infeasibility is bounded above by the size of the solution. In particular, $v^{(l+1)} = 0 \Rightarrow I(q^{(l+1)}) = 0$
- Linear $O(h)$ method yields quadratic $O(h^2)$ infeasibility
- Velocity remains bounded
- No need to change the step size to control infeasibility
- Solve one linear complementarity problem per step

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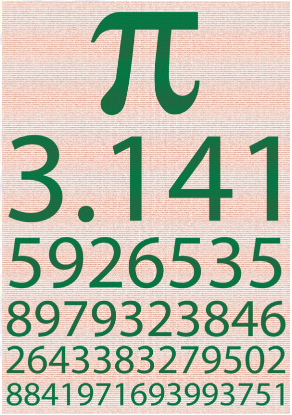
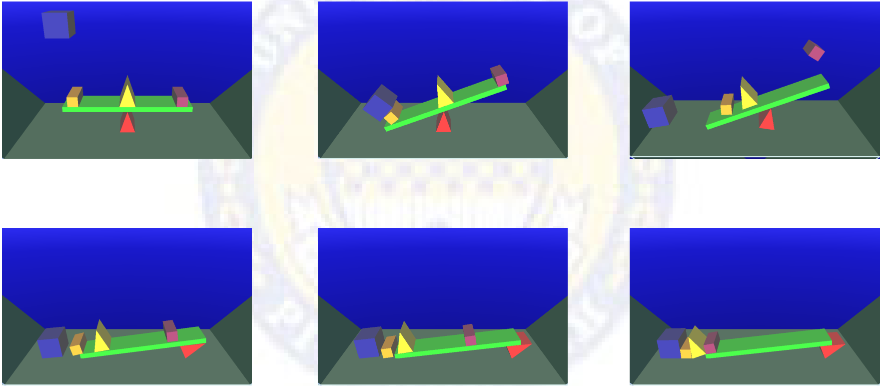
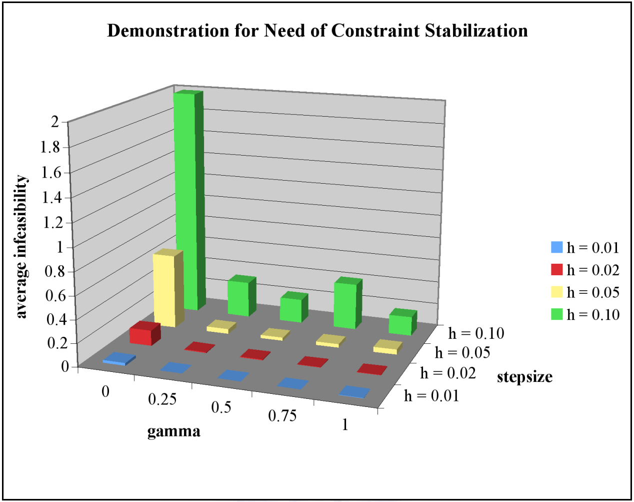


Six successive frames from Balance2



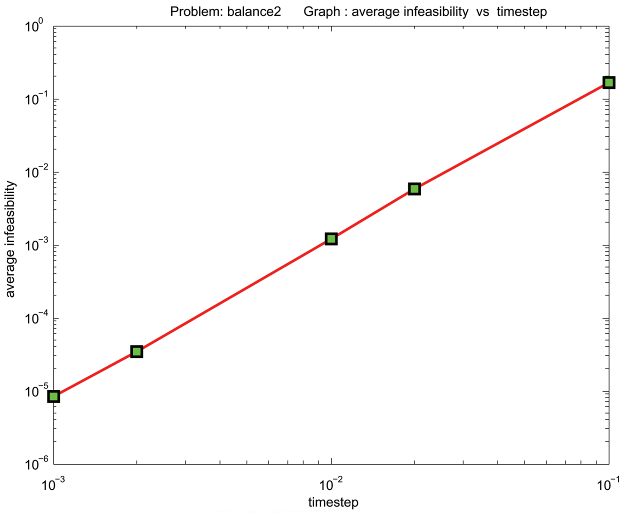
Balance2



Smaller stepsize ⇒ smaller average infeasibility

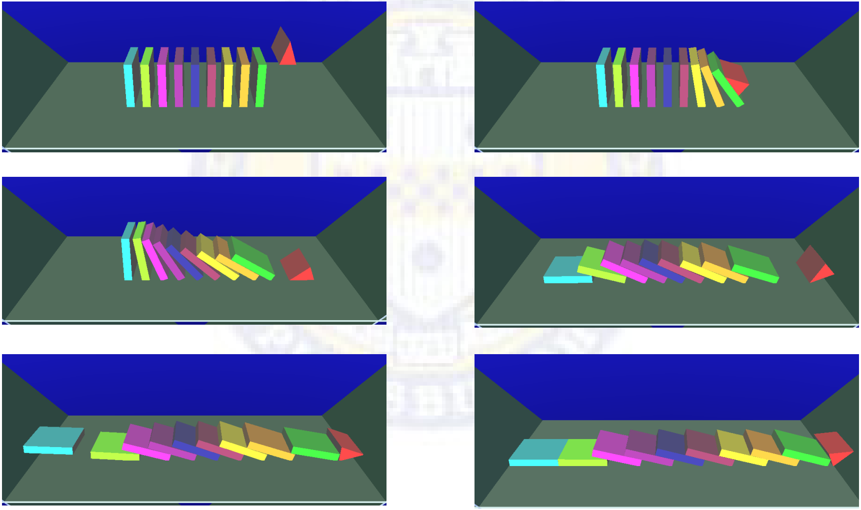
Constraint stabilization ⇒ smaller average infeasibility

Balance2



Average infeasibility shows quadratic $O(h^2)$ nature

Six successive frames from Pyramid1



Construction of a constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

- Ratio Metric
- Differentiability
- Constraints and Model
- Algorithm
- Numerical Results
- Summary

Accomplishments

- Successfully developed a computationally efficient signed distance function, Ratio Metric
- Successfully shown equivalence of RPM to MPD
- Successfully developed and analyzed algorithm that achieves constraint stabilization solving one LCP per step
- Successfully calculated generalized gradients and showed that infeasibility at step l is upper bounded by $O(||h_{l-1}||^2\,||v^{(l)}||^2)$
- Successfully implemented this algorithm for several problems with good results

Thank You!

- ICAM
- Clemson University
- University of Pittsburgh
- University of Tennessee/Knoxville
- Virginia Tech