Finite Element Analysis for a Modified Navier-Stokes-α Model

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Modified Problem

$$
\left.\begin{aligned}
u_t + (\nabla \times u) \times \overline{u} - \nu \Delta u + \nabla P = f, &\text{ in } \Omega \times (0, T] \\
\overline{u} - \alpha^2 \Delta \overline{u} + \nabla \lambda = u, &\text{ in } \Omega \times (0, T] \\
\nabla \cdot \overline{u} = \nabla \cdot u = 0, &\text{ in } \Omega \times (0, T] \\
u(x, 0) = u_0(x), &\text{ in } \Omega \\
\overline{u} = u = 0, &\text{ on } \partial\Omega \times (0, T]
\end{aligned}\right\} . \qquad (1)
$$

What can be said about finite element computations based on this regularization of the NSE?

Rotational NSE

Vector identity:

$$(u \cdot \nabla)u = (\nabla \times u) \times u - \nabla \left(\frac{1}{2}|u|^2 \right).$$

Substitute into the NSE:

$$u_t - \nu\Delta u + (\nabla \times u) \times u + \nabla \left(p - \frac{1}{2}|u|^2 \right) = f.$$

The static pressure p is replaced by a "dynamic" or "Bernoulli" pressure $P = p - \frac{1}{2}|u|^2$.

$$\Rightarrow u_t - \nu\Delta u + (\nabla \times u) \times u + \nabla P = f.$$

Classical NS-α Equations

Though generalizable to many spatial filters, the following formulation is most common.

$$\left.\begin{aligned} u_t + (\nabla \times u) \times \overline{u} - \nu \Delta u + \nabla P = f,\ \text{in } \Omega \times (0, T] \\ \overline{u} - \alpha^2 \Delta \overline{u} = u,\ \text{in } \Omega \times (0, T] \\ \nabla \cdot \overline{u} = 0,\ \text{in } \Omega \times (0, T] \\ u(x, 0) = u_0(x),\ \text{in } \Omega \\ \overline{u} = u = 0,\ \text{on } \partial\Omega \times (0, T] \end{aligned}\right\}. \tag{2}$$

Computational Problems

- FEM for the rotational form of the NSE suffers a loss of accuracy over the convective form (P under-resolved)
- FEM for (2) in the forward-backward step problem fails during non-linear solve or gives bad output
- FEM for (2) in von Karman problem fails to form a vortex street

Refer to Layton, Manica, Neda, Olshanskii and Rebholz (2008).

Modified Model

Computational problems likely arise since $\nabla \cdot u = 0$ is not imposed.
Choose a different filter:

$$\left.\begin{aligned} u_t + (\nabla \times u) \times \overline{u} - \nu \Delta u + \nabla p = f, &\ \text{ in } \Omega \times (0, T] \\ \overline{u} - \alpha^2 \Delta \overline{u} + \nabla \lambda = u, &\ \text{ in } \Omega \times (0, T] \\ \nabla \cdot \overline{u} = \nabla \cdot u = 0, &\ \text{ in } \Omega \times (0, T] \\ u(x,0) = u_0(x), &\ \text{ in } \Omega \\ \overline{u} = u = 0, &\ \text{ on } \partial\Omega \times (0, T] \end{aligned}\right\} .$$

Equivalent to (2) for periodic boundary conditions.

Boundary Conditions

Spatial filtering in the presence of boundaries with non-periodic boundary conditions on the velocity?

- Constant α holds to order α distance from $\partial\Omega$
- Near-wall resolution: let $\alpha \to 0$ near the wall
- Near-wall modeling: BC's for $\overline{u}$ derived via a closure model
- Numerical estimations of true filtered boundary values

No slip boundary conditions and constant α are considered here.

Function spaces

$$X \equiv H_0^1(\Omega)^d = \left\{ v \in H^1(\Omega)^d \text{ such that } v|_{\partial\Omega} = 0 \right\}$$

$$Q \equiv L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \text{ such that } \int_\Omega q\, dx = 0 \right\}$$

$$V = \left\{ v \in X \text{ such that } \int_\Omega q(\nabla \cdot v)\, dx = 0, \forall q \in Q \right\}$$

We use the norms

$$\|q\|_Q = \|q\|_{L^2(\Omega)} = \|q\|$$

$$\|v\|_X = \|v\|_V = \|\nabla v\|$$

Discrete spaces

Consider conforming finite element spaces $X_h \subset X$, $Q_h \subset Q$, locally quasi-uniform mesh, uniform LBB_h condition:

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{\int_\Omega q_h (\nabla \cdot v_h)\, dx}{\|q_h\| \|\nabla v_h\|} \geq \beta > 0.$$

Discretely divergence-free space:

$$V_h = \left\{ v \in X_h \, \text{such that} \int_\Omega q_h (\nabla \cdot v_h)\, dx = 0, \forall q_h \in Q_h \right\}.$$

Variational Formulation

Given $f \in H^{-1}(\Omega)$, find $(u, \overline{u}, p, \lambda) \in X \times X \times Q \times Q$ satisfying:

$$\int_\Omega \left\{ u_t \cdot v + \nu\,(\nabla u : \nabla v) + (\nabla \times u) \times \overline{u} \cdot v - p(\nabla \cdot v) \right\} dx$$

$$= \int_\Omega f \cdot v\, dx,\ \forall v \in X$$

$$\int_\Omega q(\nabla \cdot u)\, dx = 0, \forall q \in Q$$

$$\int_\Omega \left\{ \overline{u} \cdot w + \alpha^2\,(\nabla \overline{u} : \nabla w) - \lambda(\nabla \cdot w) \right\} dx = \int_\Omega u \cdot w,\ \forall w \in X$$

$$\int_\Omega r(\nabla \cdot \overline{u})\, dx = 0,\ \forall r \in Q$$

Reformulation

Equivalently, given $f \in H^{-1}(\Omega)$, find $(u, \overline{u}) \in V \times V$ satisfying:

$$\int_\Omega \left\{ u_t \cdot v + \nu \left(\nabla u : \nabla v \right) + \left(\nabla \times u \right) \times \overline{u} \cdot v \right\} dx$$
$$= \int_\Omega f \cdot v \, dx, \, \forall v \in V$$
$$\int_\Omega \left\{ \overline{u} \cdot w + \alpha^2 \left(\nabla \overline{u} : \nabla w \right) \right\} dx = \int_\Omega u \cdot w, \, \forall w \in V.$$

Discrete filter

Definition (Discrete differential filter)

Given $\phi \in L^2(\Omega)$ and $\alpha > 0$. The discrete differential filter of ϕ, $\overline{\phi}^h \in V_h$, is the unique solution to:

$$(\phi, v_h) = (\overline{\phi}^h, v_h) \; + \; \alpha^2(\nabla\overline{\phi}^h, \nabla v_h), \forall v_h \in V_h \tag{3}$$

Discrete Variational Formulation

Find $u_h, \overline{u_h}^h \in V_h$ satisfying:

$$\int_\Omega \left\{ u_{h,t} \cdot v_h + \nu(\nabla u_h : \nabla v_h) + (\nabla \times u_h) \times \overline{u_h}^h \cdot v_h \right\} dx + \gamma \int_\Omega (\nabla \cdot u_h)(\nabla \cdot v_h)\, dx = \int_\Omega f \cdot v_h\, dx, \forall v_h \in V_h \tag{4}$$

and

$$\int_\Omega \overline{u_h}^h \cdot w_h + \alpha^2 (\nabla \overline{u_h}^h : \nabla w_h)\, dx = \int_\Omega u_h \cdot w_h\, dx, \ \forall w_h \in V_h \tag{5}$$

Discrete Laplacian

Definition (Discrete Laplacian)

Given $\psi \in X$. Let $\psi_h \in V_h$ be the unique solution to:

$$\int_\Omega \psi_h \cdot v_h \, dx = -\int_\Omega \nabla \psi : \nabla v_h \, dx, \ \forall v_h \in V_h. \tag{6}$$

Then the discrete Laplacian $\Delta^h : X \to V_h$ is defined by $\Delta^h \psi = \psi_h$.

Stability

Lemma (Stability)

For a discrete solution $u_h \in V_h$, $\exists M > 0$ such that if $0 < \alpha \leq M\, h\, \nu^{1/4}$, and $0 < \gamma < \infty$, then:

$$\begin{aligned} \|\overline{u_h}^h\|^2 + \alpha^2 \|\nabla \overline{u_h}^h\|^2 + \gamma \int_0^T \|\nabla \cdot u_h\|^2\, dt \\ + \nu \int_0^T \left\{ \|\nabla \overline{u_h}^h\|^2 + \alpha^2 \|\Delta^h \overline{u_h}^h\|^2 \right\} dt \leq C \end{aligned} \tag{7}$$

$$\|u_h\|^2 + \nu \int_0^T \|\nabla u_h\|^2\, dt + \gamma \int_0^T \|\nabla \cdot u_h\|^2\, dt \leq C \tag{8}$$

where $C = C(u_0, \nu, f, \gamma, T)$ is independent of α, h.

Convergence (1)

Theorem (Convergence)

Assume $u_{NSE}(x,t)$ *is a strong solution of the NSE, satisfying*

- $\nabla \times u_{NSE} \in L^2(0, T;\ L^\infty(\Omega))$
- $\overline{u_{NSE}} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$.

with the bounds on $\overline{u_{NSE}}$ *independent of* α. *Then under the stability assumptions, the FEM approximation* $u_h \in V^h$ *converges optimally in* $L^2(0, T; H^1(\Omega))$.

Convergence (2)

- Scaling $\alpha = O(h\,\nu^{1/4})$
- Consistency of $NS - \alpha$ is order α^2, limiting FEM to $O(h^2)$
- Decreased regularity requirements on u_{NSE} possible with increased regularity of $\overline{u_{NSE}}$
- Estimates on $\overline{u_{NSE}}$ are α-independent if, for example, $u_{NSE} \in H^2$
- Taylor-Hood elements are a natural choice

Circular domain, zero BC's

A smooth, divergence free 2-D flow on the unit circle with zero boundary conditions:

$$
\begin{aligned}
u(x,y,t) &= 2^{-t}(1-x^2-y^2) < y, -x > && (9)\\
p(x,y,t) &= -\frac{1}{6}2^{-2t}\left((1-x^2-y^2)^3 - \frac{1}{4}\right) && (10)\\
f(x,y,t) &= 2^{-t}\left(\ln(2)(1-x^2-y^2) - 8\nu\right) < y, -x > && (11)
\end{aligned}
$$

All computations in FreeFem++, Taylor-Hood elements, Delaunay triangulation, two-leg Crank-Nicholson time stepping.

Using $\Delta t = 0.01$ and $\nu = 1$.

h	$\|(u-u^h)\|$	Rate	$\|\nabla(u-u^h)\|$	Rate
0.393	1.22e-2	—-	1.26e-1	—-
0.191	2.98e-3	2.07	4.05e-2	1.67
0.101	7.32e-4	1.96	1.33e-2	1.55
0.051	1.81e-4	2.32	4.53e-3	1.79
0.030	4.96e-5	2.37	1.71e-3	1.79

Table: **L^2 and H^1 errors and rates for circular flow.**

Square domain, zero BC's

Using a square domain where boundary approximation is exact. Chosen solution:

$$
\begin{aligned}
u_1(x,y,t) &= x^2(x-1)^2(2y^3-3y^2+y) \\
u_2(x,y,t) &= -y^2(y-1)^2(2x^3-3x^2+x) \\
p(x,y,t) &= 0
\end{aligned}
$$

$\nu = 10^{-3}$ with $\Delta t = 5 \cdot 10^{-3}$.

h	$\|u-u^h\|$	Rate	$\|\nabla(u-u^h)\|$	Rate
1.088e-1	9.620e-7	—	4.141e-4	—
5.657e-2	7.225e-8	3.96	1.131e-4	1.98
2.886e-2	5.096e-9	3.94	2.952e-5	2.00
1.458e-2	4.845e-10	3.45	7.539e-6	2.00

Table: **Errors and convergence rates, square domain, zero BC's.**

Taylor-Green vortices

Using Taylor-Green vortices on the unit square as a true solution. Driving force $f = 0$,

$$u_1(x, y, t) = -cos(N\pi x)sin(N\pi y)e^{-2N^2\pi^2\nu t}$$
$$u_2(x, y, t) = cos(N\pi y)sin(N\pi x)e^{-2N^2\pi^2\nu t}$$
$$p(x, y, t) = -\frac{1}{4}cos(2N\pi x)cos(2N\pi y)e^{-2N^2\pi^2\nu t}.$$

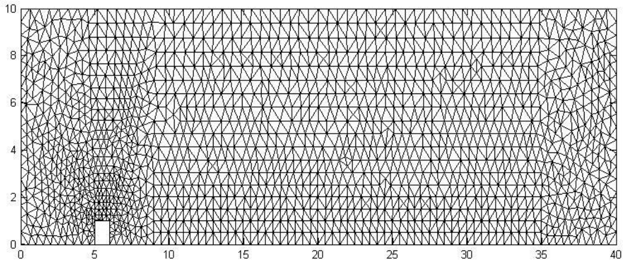
We choose $N = 2$. The viscosity parameter is $\nu = 10^{-2}$, $\Delta t = 5 \cdot 10^{-3}$.

h	$\|u - u^h\|$	Rate	$\|\nabla(u - u^h)\|$	Rate
1.28565e-1	5.94047e-2	—-	9.51165e-1	—-
6.73435e-2	1.87157e-2	1.786	3.16293e-1	1.703
3.44930e-2	4.20793e-3	2.231	7.64096e-2	2.123
1.74594e-2	9.45320e-4	2.193	1.77568e-2	2.143

Table: **Errors and convergence rates, periodic boundary conditions.**

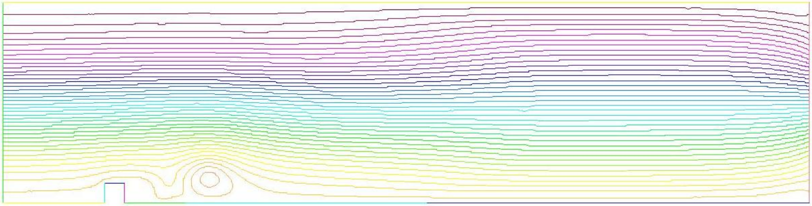
Forward-backward step

Non-uniform mesh, refined near the step. Calculations for $\nu = 1/600$, $\Delta t = 0.005$.
Boundary conditions $u_{in} = u_{out} =< \frac{1}{25}y(10-y), 0 >$, $u = 0$ on top/bottom.



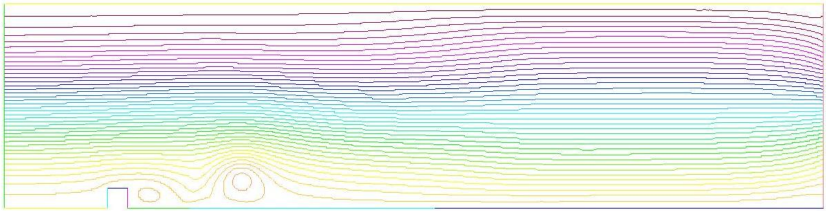
NS-α

Streamline plot for NS-α.



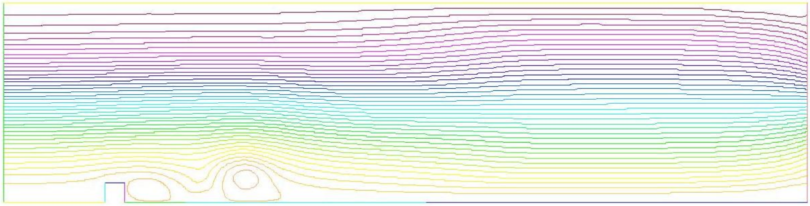
Rotational NSE

Streamline plot for NSE.



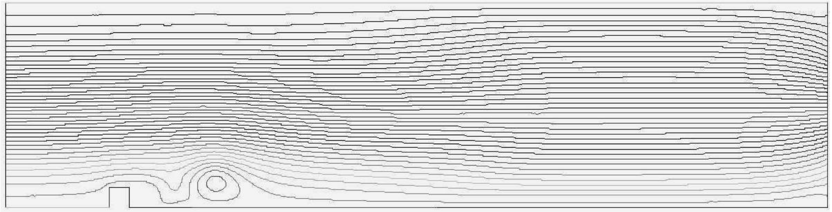
Convective NSE

Streamline plot for NSE.



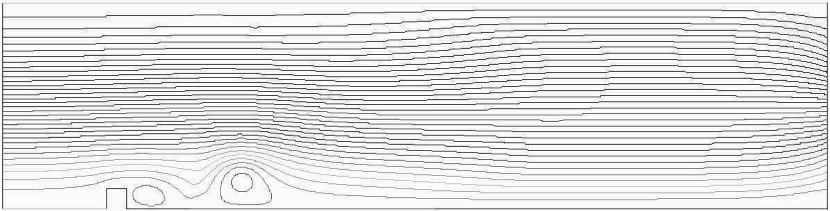
NS-α

Streamline plot for NS-α.



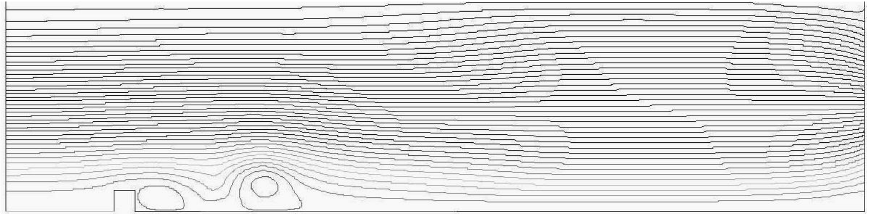
Rotational NSE

Streamline plot for NSE.



Convective NSE

Streamline plot for NSE.



Other work

- Grad-div stabilization not thoroughly explored
- Smaller meshes and time steps
- Boundary conditions
- Large scale problems (geophysical flow)
- Other regularizations and ADM's?