# Optimal control applied to a model of species augmentation

Erin N. Bodine

Co-authors: Suzanne Lenhart & Louis Gross

The University of Tennessee Knoxville

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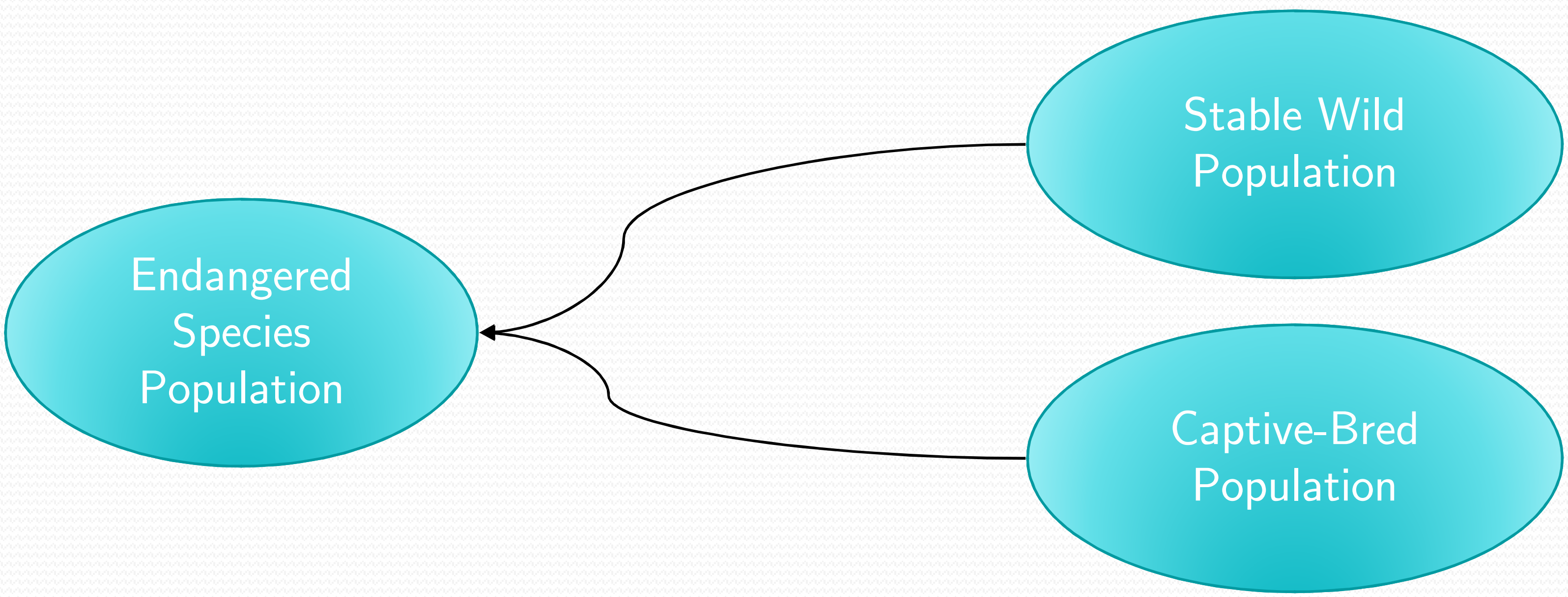
# Outline

- Introduction to Species Augmentation
- Formulating the Optimal Control Model
- Numerical Simulations & Results
- Work in Progress – A Discrete Time Version
- Questions

# Species Augmentation

Species augmentation is a method of reducing species loss by augmenting a declining or threatened population with individuals from captive-bred or stable, wild populations.



# Where has species augmentation already been used?

1995 Florida panther (*Felis concolor coryi*) augmentation

Eight female Texas panthers (another panther subspecies) were placed in the Southern Florida panther ranger in an effort to raise the population size and increase genetic diversity

**30 – 50  panthers in 1995**
**80 – 100 panthers in 2007**

# Where are the models?

- There currently do not exist *general* mathematical species augmentation models to address questions concerning the dynamics of augmented populations and communities.
- My dissertation research aim is to begin developing a mathematical framework to address the biological questions surrounding species augmentation.
- As a general augmentation theory is developed, the results can be readily adapted to a variety of specific cases.

# Our Model

- The first model we have developed a continuous time model that addresses the question

*At what rate does the target population need to be augmented to meet certain objectives?*

- *Mathematical Biosciences & Engineering* 2008, 5(4): 669 - 680.

# Simple Two-Population Model

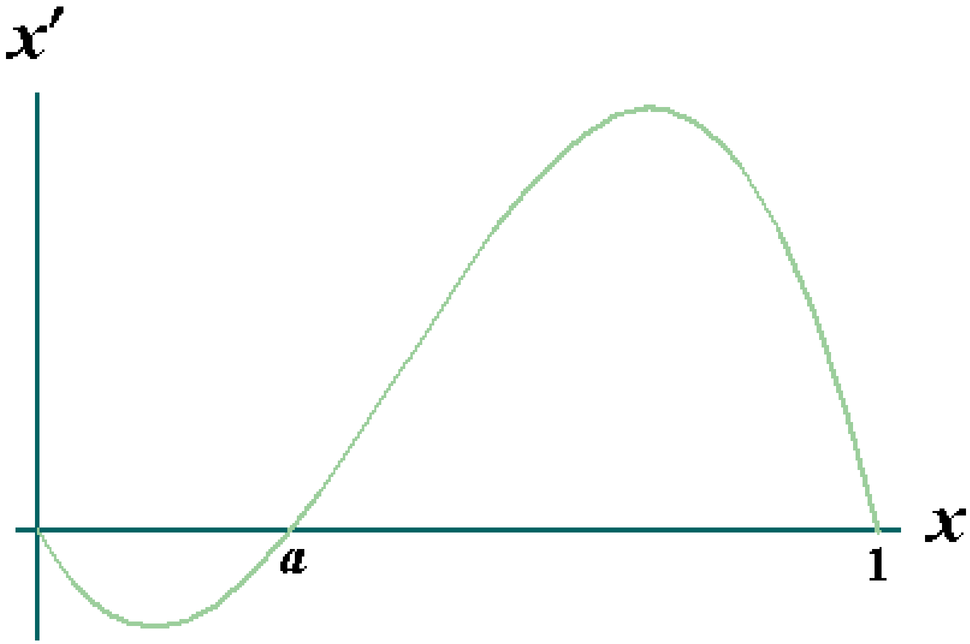
- Start with two populations
  - $x$ the target/endangered population
  - $y$ the reserve population

- Assume both populations grow according to a simple population growth model with *Allee effect* in the absence of human intervention.

# Allee Effect Model

**Normalized Allee Equation**

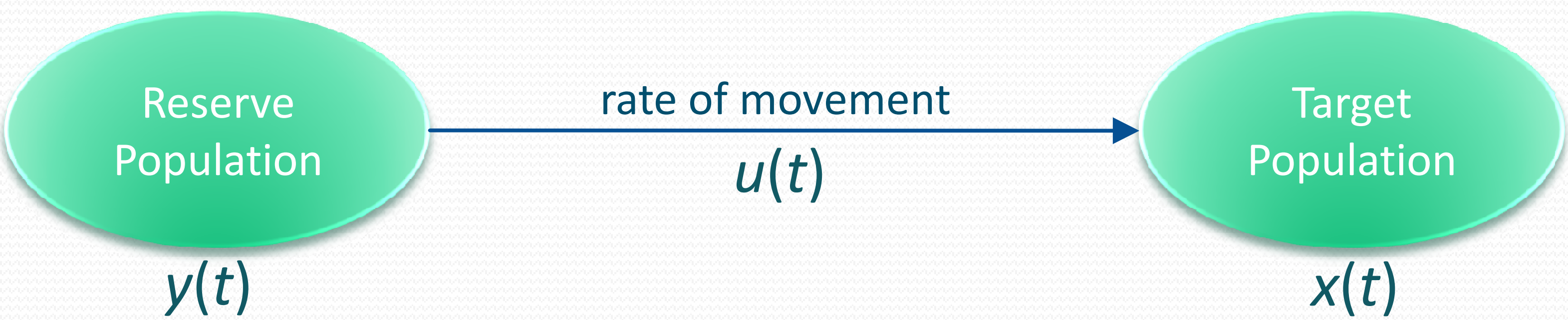
$$\frac{dx}{dt} = rx(t)\left(1 - x(t)\right)\left(x(t) - a\right)$$




| | |
|---|---|
| *x*(*t*) | Population size at time *t* |
| *r* | Intrinsic growth rate |
| *a* | Minimum population density for growth |

# Model Assumptions

We wish to move individuals from *y* to *x* over a period $[t_0, t_1]$ of time such that *x* is as large as possible by the end of the time period.



# Model Assumptions

- Target population starts below minimum threshold for growth, $0 \leq x(t_0) < a$, so population is declining
- Reserve population starts above minimum threshold for growth, $b \leq y(t_0) < 1$ , so population is growing
- In addition to maximizing $x$ by the final time, do not want to completely deplete $y$
- Minimize cost of harvesting/augmenting

# Optimal Control Formulation

The optimal control formulation of this problem is:

$$\max_{0 \leq u \leq 1} \left[ x(t_1) + By(t_1) - A \int_{t_0}^{t_1} u^2(t)dt \right]$$

$$\begin{aligned} x'(t) &= rx(1-x)(x-a) + puy, \quad x(t_0) = x_0 < a \\ y'(t) &= sy(1-y)(y-b) - uy, \quad y(t_0) = y_0 > b \end{aligned}$$

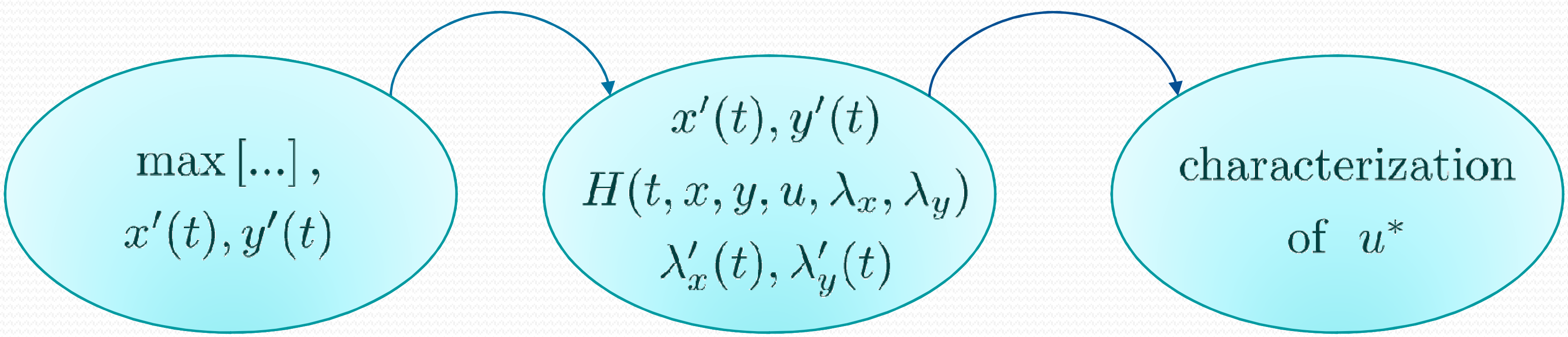
The parameter *p* is the ratio of the reserve population carrying capacity to the target population carrying capacity.

# Using Pontryagin's Maximum Principle

Pontryagin's Maximum Principle

- provides necessary conditions for a control to be optimal
- applying Pontryagin's Maximum Principle we obtain a characterization of the optimal control

Pontryagin's Maximum Principle



# Characterization of Optimal Control

Construct the Hamiltonian *H*

$$H = -Au^2 + \lambda_x(rx(1-x)(x-a) + puy) + \lambda_y(sy(1-y)(y-b) - uy)$$

and the adjoint functions $\lambda_x$ and $\lambda_y$ corresponding to the states *x* and *y*

$$\begin{aligned}\lambda_x' &= -\frac{\partial H}{\partial x} = \lambda_x r\left(3(x^*)^2 - 2x^*(1+a) + a\right), \quad \lambda_x(t_1) = 1 \\ \lambda_y' &= -\frac{\partial H}{\partial y} = \lambda_y s\left(3(y^*)^2 - 2y^*(b+1) + b\right) - \lambda_x p u^* + \lambda_y u^*, \quad \lambda_y(t_1) = B\end{aligned}$$

Maximize *H* with respect to *u* to get the optimal control $u^*(t)$

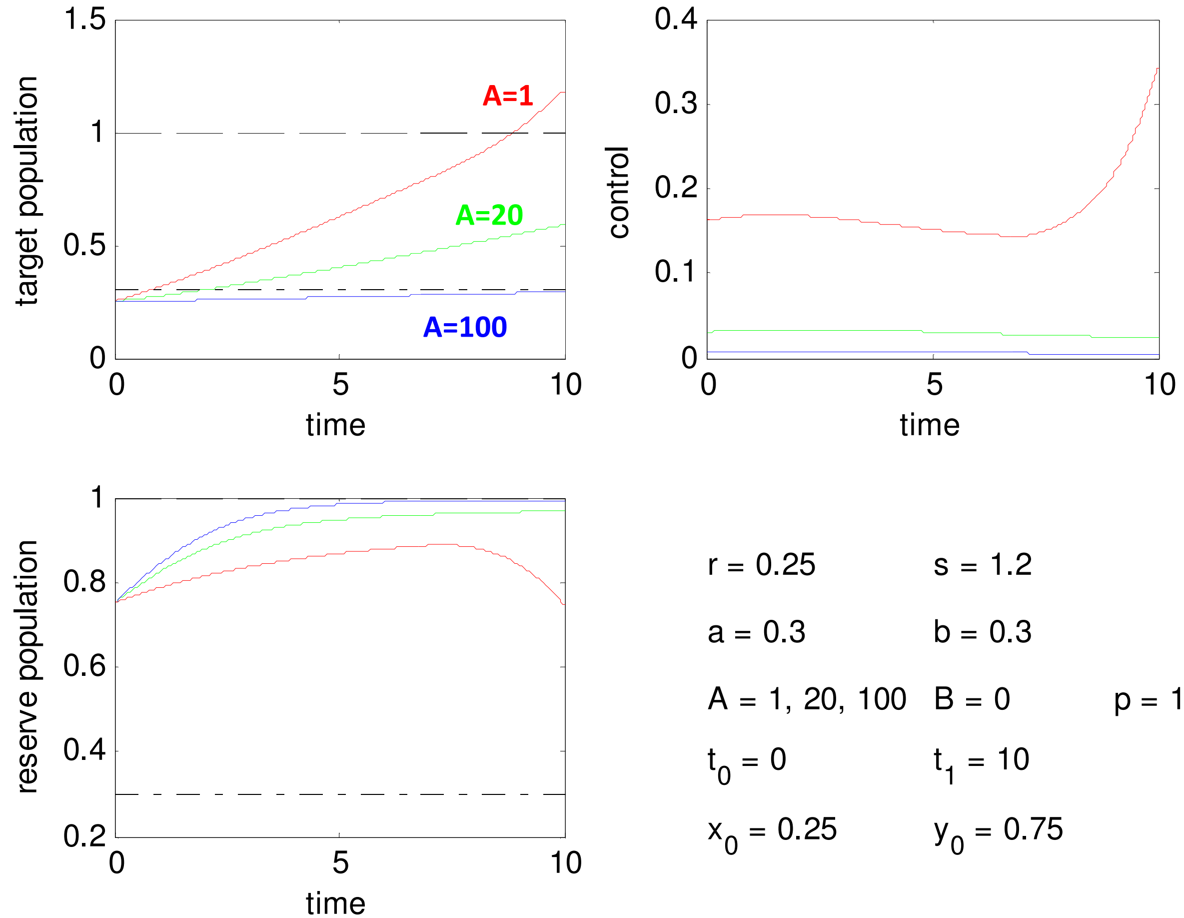
$$u^*(t) = \min\left\{1, \max\left\{0, \frac{p\lambda_x(t) - \lambda_y(t)}{2A} y^*(t)\right\}\right\}$$

# Numerical Simulations

- Make an initial guess for $u^*(t)$
- Solve state equations (with initial conditions) using Runge-Kutta 4 method
- Solve adjoint equations (with terminal conditions) using Runge-Kutta 4 method
- Update $u^*(t)$ using the characterization of the optimal control
- Repeat until meet convergence condition

# Results

## Varying cost coefficient of translocation




$r = 0.25$ $s = 1.2$

$a = 0.3$ $b = 0.3$

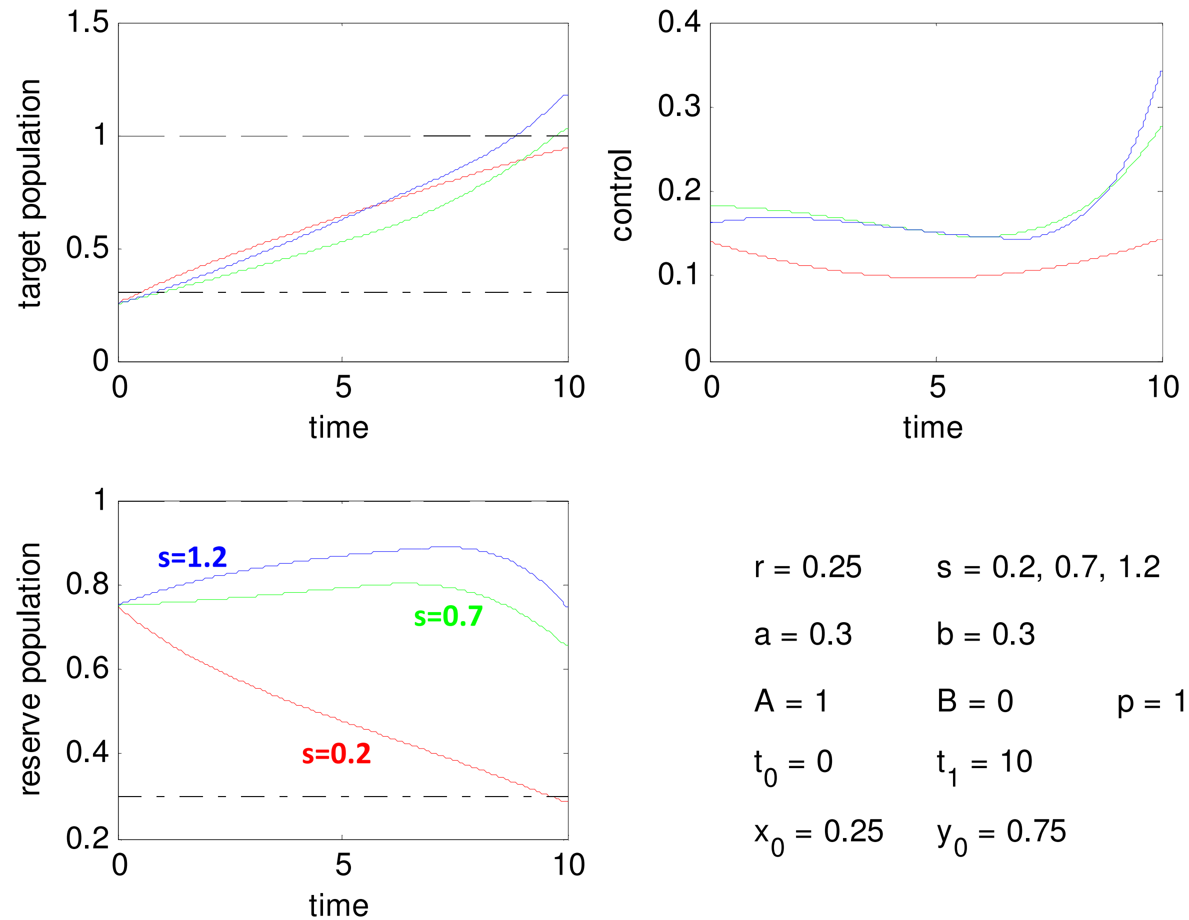
$A = 1, 20, 100$ $B = 0$ $p = 1$

$t_0 = 0$ $t_1 = 10$

$x_0 = 0.25$ $y_0 = 0.75$

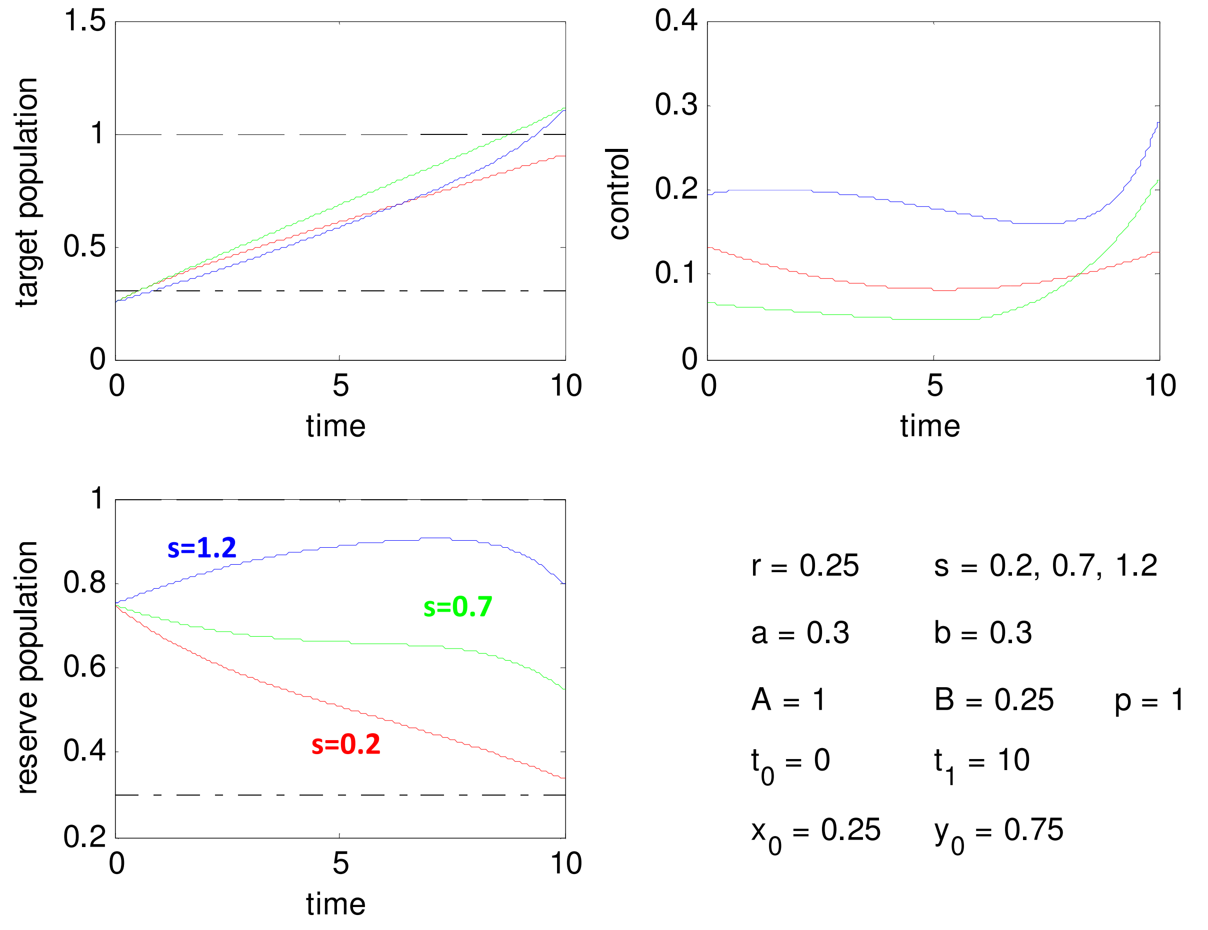
# Results

## Varying ratio of intrinsic growth rates, with low cost of translocation



# Results

As before but increasing importance of maximizing reserve pop by final time



# Conclusions

- High cost can prevent moving enough individuals to the target population so that it is above its threshold for population growth
- Low cost can lead to an “over-augmenting” of the target population. Augmenting a population above its carrying capacity results in wasted resources.
- The combination of a low cost of translocation and a low intrinsic growth rate for the reserve population could lead to the reserve population falling below its threshold for population growth by the final time.
- This can be counteracted by increasing the importance of having a large reserve population by the final time, i.e. increase the $B$ value

# Future Augmentation Modeling

- Discrete Time
  - Augmentation usually occurs as a single or a few translocations of individuals
  - We are working on creating a discrete time version of this model to compare with these continuous time results
- Linearity of the control in the cost term
  - Model presented uses a control which is quadratic in the cost term
  - Can be interpreted as "as augmentation rate (*u*(*t*)) increases, the rate of increase of the cost increases"

$\int_{t_0}^{t_1} u(s)ds$ instead of $\int_{t_0}^{t_1} u^2(s)ds$

# Thank you!