Mean Motions and Longitudes in Indian Astronomy

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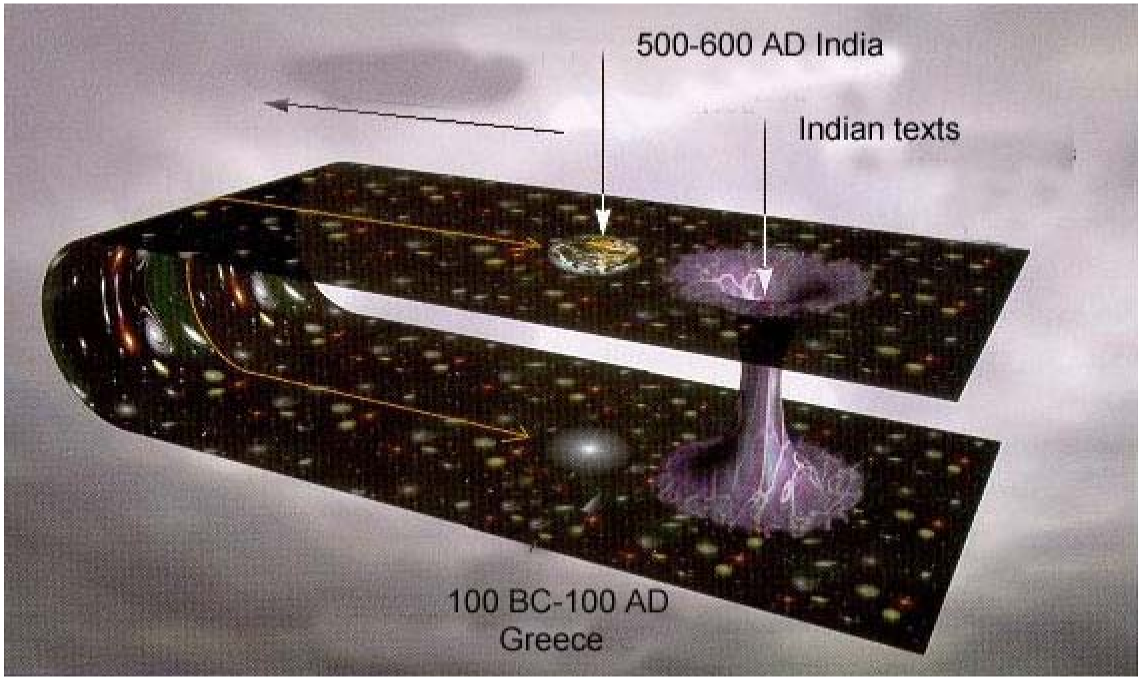
In many ways ancient Greek and Indian astronomy are very similar. This has led to what can be called the Neugebauer–Pingree–van der Waerden Hypothesis (NPvdW):

> *The texts of ancient Indian astronomy give us a sort of wormhole through space-time back into an otherwise inaccessible era of Greco-Roman developments in astronomy.*



However, in one otherwise elementary and relatively simple area – the expression of **mean motions** and **mean longitudes** – the Greek and Indian traditions look very different.

In the *Almagest* mean motions and longitudes are in the form

$$\bar{\lambda} = \bar{\lambda}_0 + \omega(t - t_0)$$

but in Indian texts we always see things like

$$\bar{\lambda} = \frac{R}{Y}(t - t_0) = R' + r' \text{ in rotations}$$
$$= 360^\circ \times r' \text{ in degrees}$$

Typical values are t_0 = –3101 Feb 18 at 6 AM – 1,972,944,000 years
Y = 4,320,000 or 4,320,000,000
R = 2,296,824 or 2,296,828,522 (Mars)

Let's work an example for Saturn: $R = 146{,}564^r$ and $Y = 4{,}320{,}000^y$.
As far as we know these are *sidereal* revolutions and years.

Clearly the mean longitude of Saturn and every other planet satisfies

$$\bar{\lambda} = 0° \text{ at } t = t_0 \text{ and again at } t = t_0 + nY, \quad n = -\infty,\ldots,+\infty$$

thus there is an infinite cycle of grand mean conjunctions every $4{,}320{,}000^y$.

If we take $t_0 = -3101$ Feb 18 at 6 AM and $t - t_0 = 3600^y$ we get

$$\bar{\lambda} = \frac{146{,}564}{4{,}320{,}000} 3600 = 122^r + \frac{41^r}{300}$$

Dropping the 122 complete revolutions, after 3600^y the mean longitude of Saturn is

$$\bar{\lambda} = \frac{41}{300} \times 360^\circ = 49;12^\circ$$

In fact, the mean *tropical* longitude at that time is 48;56°, just 0;16° different.

If you repeat this exercise for the other planets you find

		modern	Sunrise	Almagest
Mercury	499	183.42	186.00	178.17
Venus	499	356.23	356.40	351.14
Sun	499	359.76	0.00	357.18
Mars	499	6.91	7.20	4.36
Jupiter	499	186.85	187.20	185.35
Saturn	499	48.93	49.20	45.93
Moon	499	280.66	280.80	278.62
apogee	499	35.44	35.70	32.40
node	499	352.01	352.20	349.09

If you adjust for the small 0;14° offset in the vernal equinox you find the following differences between modern (tropical) theory and Hindu (sidereal) theory of about A.D. 500:

		(arcmin)
Mercury	499	141
Venus	499	-4
Sun	499	0
Mars	499	3
Jupiter	499	7
Saturn	499	2
Moon	499	-6
apogee	499	1
Node	499	-3

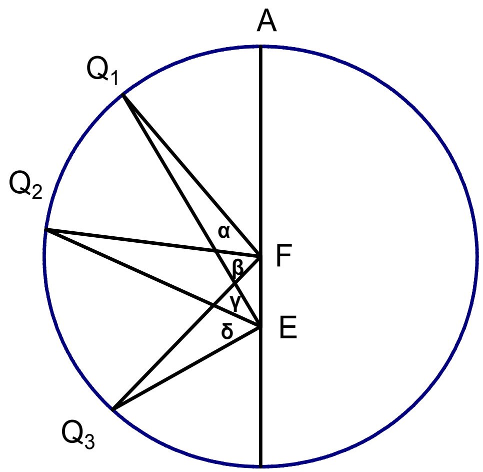
Except for Mercury this is comparable to the accuracy of Tycho Brahe in 1580. How does this work? Where does it come from?

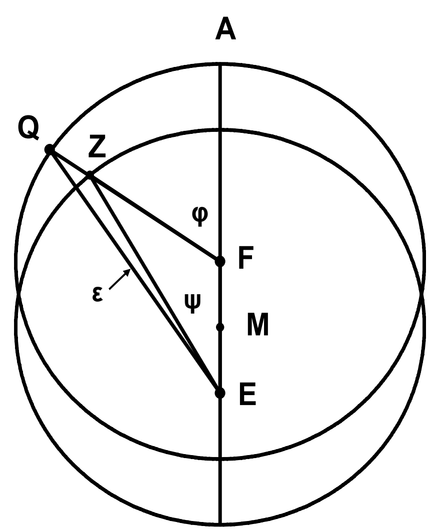
Almagest review

the **mean motions** come from empirical data or period relations with small corrections:

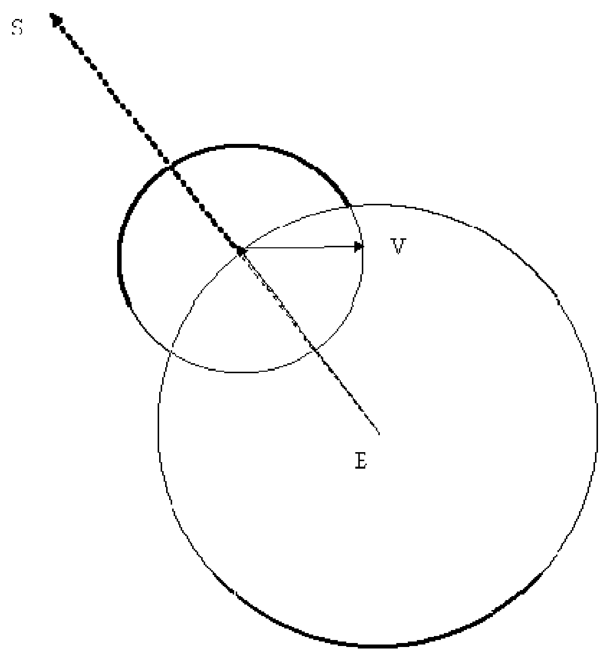
e.g. Mars: $\omega = \dfrac{192^r + 61;43^\circ}{410^{ey} + 231;40^d}$ or $\omega = \dfrac{37^r}{79^{ty} + 3;13^d}$

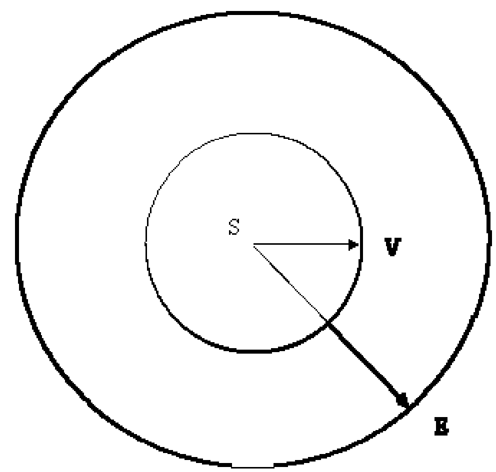
the **mean longitudes** at some moment come from complicated analyses of mean oppositions or mean greatest elongations, which also give $\boldsymbol{\lambda_A}$ and ***e***.





Furthermore, there are clear relations between the mean motion of the Sun and the mean motions in longitude and anomaly of the planets:

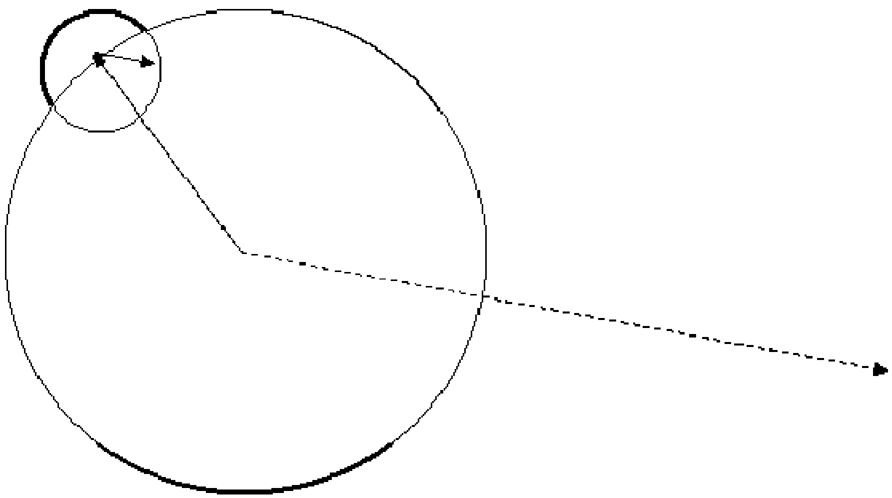


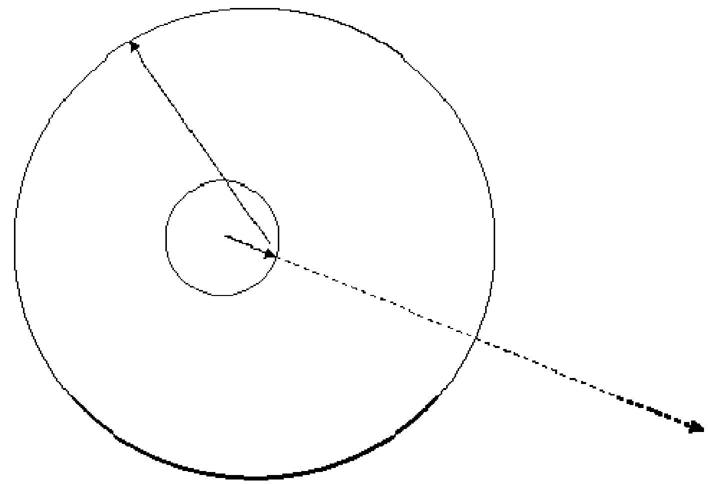


For the inner planets, not only does the planet revolve on an epicycle, but the center of the epicycle is always lined up with the Sun:

$\omega_p = \omega_S$ or $\omega_p = \omega_S + \omega_a$ http://www.csit.fsu.edu/~dduke/venhelio.html

For the outer planets, the radius of the epicycle is always parallel to the direction of the Sun from the earth: $\omega_p = \omega_S - \omega_a$





http://www.csit.fsu.edu/~dduke/juphelio.html

The numbers we need to understand are the revolutions in 4,320,000^{y}. These are

	Revolutions /mahayuga	Revolutions /kalpa
Mercury	17,937,020	17,936,998,984
Venus	7,022,388	7,022,389,522
Sun	4,320,000	4,320,000,000
Mars	2,296,824	2,296,828,522
Jupiter	364,224	364,226,455
Saturn	146,564	146,567,298
Moon	57,753,336	57,753,300,000
apogee	488,219	488,105,858
node	–232,226	–232,311,168

For Mercury and Venus those are revolutions of the epicycle with respect to a fixed (sidereal) direction, and this is not an invention of Indian astronomy.

In the *Almagest* Ptolemy uses ω_a and $\omega_p = \omega_S$ for the inner planets, and in the *Planetary Hypotheses* he gives also $\omega_p = \omega_S + \omega_a$ in sidereal (not tropical) terms, exactly as we find in all the Hindu texts:

Mercury $\quad \omega_p = \dfrac{865^r + 0;4^\circ}{208^{ey} + 174^d}$ on the epicycle

$$\omega_p = \omega_S + \omega_a$$

Venus $\quad \omega_p = \dfrac{57^r + 0;1^\circ}{35^{ey} + 33^d}$ on the epicycle

Mars $\quad \omega_p = \dfrac{51^r - 0;3^\circ}{95^{ey} + 361^d}$ on the deferent

Jupiter $\quad \omega_p = \dfrac{18^r + 0;1^\circ}{213^{ey} + 240^d}$ on the deferent

$$\omega_p = \omega_S - \omega_a$$

Saturn $\quad \omega_p = \dfrac{4^r + 0;1^\circ}{117^{ey} + 330^d}$ on the deferent

The same scheme is also seen in the *Keskintos Inscription* (*ca.* 100 B.C., see Alexander Jones' *SCIAMVS* article).

The Indian texts use very long time intervals: $4{,}320{,}000{,}000^y$ and $4{,}320{,}000^y$

The long time intervals in *Paita* and *BSS* have a rather intricate structure:

1 *kalpa* = $4{,}320{,}000{,}000^y$
= 14 *manvantaras* ($306{,}720{,}000^y$ each) + 15 *sandhis* ($1{,}728{,}000^y$ each)
= 1,000 *mahayugas*

1 *manvantara* = 71 *mahayugas* ($4{,}320{,}000^y$ each)

1 *mahayuga* = 1 *krtayuga* ($1{,}728{,}000^y$) +1 *tretayuga* ($1{,}296{,}000^y$) +
1 *dvaparayuga* ($864{,}000^y$) + 1 *kaliyuga* $432{,}000^y$)
(note 4:3:2:1 ratios)

Finally, the epic battle of Bharata occurred at the beginning of the *kaliyuga* of the 28th *mahayuga* of the 6th *manvantara* of the current *kalpa*.

Note that all numbers are multiples of $432{,}000^y$.

The time from the beginning of the current *kalpa* to the epic *kaliyuga* turns out to be 4,567 periods of $432{,}000^y$, or $1{,}972{,}944{,}000^y = 0.4567$ *kalpas.* This number 4,567 will play a major role in the following.

Aryabhata uses a similar scheme but it differs in details. He dispenses with the intervening twilight periods (*sandhis*) and makes

1 *kalpa* = 1,008 *mahayugas* ($4{,}320{,}000^y$ each)
= 14 *manvantaras* (72 *mahayugas* each)
= $4{,}354{,}560{,}000^y$

1 *mahayuga* = 4 *kaliyugas* ($1{,}080{,}000^y$ each) (note 1:1:1:1 ratios)

Exactly as before, the epic battle of Bharata occurred at the beginning of the final *kaliyuga* of the 28th *mahayuga* of the 6th *manvantara* of the current *kalpa*.

For Aryabhata, the time from the beginning of the current *kalpa* to the epic *kaliyuga* turns out to be $1{,}986{,}120{,}000^y \approx 0.4561$ *kalpas*, but as it turns out, the number 4,561 will play no role whatsoever for Aryabhata.

All of this is explained in the ancient (100 B.C. – A.D. 400) literary tradition of the Hindus and is there is no evidence of any astronomical influence.

The place where there very likely *is* astronomical influence is placing the date of the beginning of the critical final *kaliyuga* in relation to some modern date. This date turns out to be

–3101 Feb 18 at sunrise (6 AM) (*Paita*, *BSS*, and *Aryabhatiya*)

–3103 Feb 17/18 at midnight (*Aryasiddhanta*)

This date is related to the date of the Great Deluge (flood of Noah, etc), and its exact origin, whether Hindu, Persian, Greek, Babylonian, etc, has been a subject of intense debate (e.g. Pingree vs. van der Waerden, 1950's – 1990's).

In the long period, *Paitamahasiddhanta* (anon.) and Brahmasphutasiddhanta (Brahmagupta), the year length is $365;15,30,22,30^d$.

In the short period the year length is $365;15,31,15^d$ (sunrise) or $365;15,31,30^d$ (midnight), both by Aryabhata.

Let's begin with the short period systems (they are easiest). Aryabhata clearly implies a modern date $3{,}600^y$ after the date of the battle. Since that date is $1/4^d$ earlier in the midnight system and the midnight year-length is $0;0,0,15^d$ longer, an interval of 3600^y in *both* systems brings us to exactly the same date:

499 Mar 21 at noon.

This date ***just happens*(?!)** to be very near a true vernal equinox ($\lambda_S = 359;46°$), i.e. just within the typical $¼^d$ uncertainty in cardinal dates known since at least the time of Hipparchus.

Aryabhata's scheme obviously leads to a mean conjunction of all the planets at the beginning and end of each *mahayuga*, but he adds the further condition that a mean conjunction must also occur at the beginning of the final *kaliyuga*.

This then requires that **each *R* be divisible by four**, and hence there is in fact a mean conjunction at the beginning of each of the four equal *kaliyugas*, or every 1,080,000 years.

The lunar apogee and node are special cases. The apogee is set to 90° and the node to 180° at the beginning of the final *kaliyuga*. This then requires that

R for the **apogee** must be of the form **three plus a multiple of four**,

R for the **node** must be of the form **two plus a multiple of four**.

For 499 Mar 21 at noon we need $\bar{\lambda} = 3{,}600 \times \frac{R}{L} = 3{,}600 \times \omega$.

Since 3,600 is sexagesimally expressed as 1,0,0 the calculation is most transparent if we also express ω in a sexagesimal base.

Since R must be an integer divisible by 4, ω is of the form $\omega = n;\alpha,\beta,\gamma,\delta$ where δ must be either 0, 12, 24, 36, or 48. Then we can find $\bar{\lambda}$ by a simple double left-shift:

$$\bar{\lambda} = \omega(t - t_0) = 1,0,0 \times \omega = R' + r' = n,\alpha,\beta;\gamma,\delta$$

so $R' = n,\alpha,\beta$ revolutions and $r' = 0;\gamma,\delta$ revolutions, or, in degrees,

$$\bar{\lambda} = 360^\circ \times 0;\gamma,\delta = 6^\circ \times \gamma;\delta.$$

Clearly, $\bar{\lambda}$ will be an integral multiple of 1/300$^{\text{th}}$ revolution or 1;12°, and if R changes by ±4 revolutions, then $\bar{\lambda}$ changes correspondingly by ±1;12°.

We can now apply this case by case:

Saturn $\omega = \frac{146,564}{4,320,000} = 0;2,2,8,12$

$6^\circ \times 8;12 = 49;12^\circ$ and $\frac{59-57}{59} = \frac{2}{59} = 0;2,2,2....$

Jupiter $\omega = \frac{364,224}{4,320,000} = 0;5,3,31,12$

$6^\circ \times 31;12 = 187;12^\circ$ and $\frac{83-76}{83} = \frac{7}{83} = 0;5,3,36...$

Mars $\omega = \frac{2,296,824}{4,320,000} = 0;31,54,1,12$

$6 \times 1;12 = 7;12^\circ$ and $\frac{284-133}{284} = \frac{151}{284} = 0;31,54,5...$

Venus $$\omega = \frac{7,022,388}{4,320,000} = 1;37,31,59,24$$

$$6^\circ \times 59;24 = 356;24^\circ \text{ and } \frac{152+243}{243} = \frac{395}{243} = 1;37,31$$

Mercury $$\omega = \frac{17,937,020}{4,320,000} = 4;9,7,31,0$$

$$6^\circ \times 31;0 = 186^\circ \text{ and } \frac{46+145}{46} = \frac{191}{46} = 4;9,7,49$$

the Moon $$\omega = \frac{57,753,336}{4,320,000} = 13;22,7,46,48$$

$$6^\circ \times 46;48 = 280;48^\circ, \text{ and } \frac{235+19}{19} = \frac{254}{19} = 13;22,6$$

the Moon's apogee $\omega = \dfrac{488,219}{4,320,000} = 0;6,46,50,57$

$$90^\circ + 6^\circ \times 50;57 = 35;42^\circ, \text{ and } \frac{254}{19} - \frac{235}{19} \times \frac{269}{251} = 0;6,46$$

the Moon's node $\omega = \dfrac{-232,226}{4,320,000} = -0;3,13,31,18$

$$180^\circ - 6^\circ \times 31;18 = -7;48^\circ, \text{ and } \frac{254}{19} - \frac{235}{19} \times \frac{5923}{5458} = -0;3,13$$

Clearly the revolution numbers R contain exactly **two, and *only* two**, components: (1) knowledge of widely known planetary period relations, and (2) rounded mean longitudes at noon on 499 Mar 21. These deconstructions of the R's clearly suggest one way the numbers might have been originally constructed: from the period relations one would compute $n;\alpha,\beta$, from the mean longitudes, rounded to the nearest multiple of 1;12°, one would get γ,δ, and thus compute R as

$$R = 4,320,000 \times n;\alpha,\beta,\gamma,\delta = 20,0,0 \times n;\alpha,\beta,\gamma,\delta$$

Also note that because of the required rounding,

we know only intervals of width 1;12° that the author thought the mean longitudes were in on 499 Mar 21, and so

we have **no information where in those intervals** his assumed mean longitudes might lie.

499 Mar 21 noon	Almagest	Sunrise
Mercury	−2;47	(1;45,2;57)
Venus	−2;32	(-0;40,0;32)
Sun	0;00	0
Mars	0;03	(-0;33,0;39)
Jupiter	1;07	(-0;29,0;43)
Saturn	0;23	(-0;34,0;38)
Moon	−0;04	(-0;42,0;30)
apogee	0;25	(-0;35,0;37)
node	−0;18	(-0;39,0;33)

For the long period system the fundamental period is 4,320,000,000 years, or 1,000 times longer.

Not only are the mean longitudes zero at the beginning and end of each *kalpa*, but so also are the apogees and nodes of all the planets, and so the beginning and end of **each *kalpa* is a true conjunction** of all the planets, not just a mean conjunction.

The beginning of the current *kaliyuga* is said to occur when $0.4567 \times 4,320,000,000 = 1,972,944,000$ years have passed.

The number R of revolutions in 4,320,000,000 years is adjusted so that
(a) the mean longitude lies in a desired interval at some contemporary date, and
(b) the mean longitudes are all near, but not exactly equal to, 0° at the beginning of the current *kaliyuga*. The number of days in 4,320,000,000 years is said to be 1,577,916,450,000, so the sidereal year length is 365;15,30,22,30, and 3,600 of these years brings us to **499 Mar 20 at 3 PM, or 21 hours earlier** than Aryabhata's modern date.

The numbers we need to understand are the revolutions in 4,320,000^y. These are

	Revolutions /mahayuga	Revolutions /kalpa
Mercury	17,937,020	17,936,998,984
Venus	7,022,388	7,022,389,522
Sun	4,320,000	4,320,000,000
Mars	2,296,824	2,296,828,522
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For Mercury and Venus those are revolutions of the epicycle with respect to a fixed (sidereal) direction, and this is not an invention of Indian astronomy.

Saturn as an example: we might expect that R will be about

$$2/59 \times 4{,}320{,}000{,}000 \simeq 146{,}440{,}678.$$

However, using that number of revolutions, after 1,972,944,000 years Saturn will have made some 66,879,457.6426 revolutions, and so have a mean longitude of about $360^\circ \times 0.6426 = 213;20^\circ$ on –3101 Feb 18, very far from the target value of 0°.

Furthermore, after an additional 3,600 years Saturn will have made an additional 122.0339 revolutions and so have a mean longitude of about 243.32°, while, recalling that Aryabhata put it in the interval (48;36°, 49;48°), we know that the mean longitude should be about 49°.

Clearly we must adjust the number of revolutions, and the first thing to do is to add enough revolutions to 146,440,678 to eliminate the fractional part of the revolutions at –3101 Feb 18.

To this end, let ρ = 146,440,678 and let us imagine adding p revolutions to ρ so that $0.4567(\rho + p)$ is some integer m. We already know that 0.4567ρ is an integer n = 66,879,457 plus a fractional part c = 0.6426, so altogether we must have

$$\begin{aligned} m &= 0.4567(\rho + p) \\ &= n + c + 0.4567\,p \end{aligned}$$

or

$$\begin{aligned} q &= m - n \\ &= c + 0.4567\,p \\ &= 0.6426 + 0.4567\,p \end{aligned}$$

Multiplying through by 10,000 we see that we must solve the indeterminate equation

$$10,000q - 4567p = 6426.$$

Greek and Indian mathematicians were certainly able to solve indeterminate equations such as

$$aq - bp = c$$

where *a, b,* and *c* are given integers. One method is to expand the rational fraction *a/b* in **a continued fraction series**, which is in itself a simple extension of Euclid's algorithm. The resulting series of convergents will of course terminate in *a/b* itself, and **the penultimate convergent, *p/q*, will be the solution** of

$$aq - bp = \pm 1.$$

Letting p_0, q_0 be the particular solution with smallest positive values of p, q, general solutions may be found from

$$p = p_0 + at$$
$$q = q_0 + bt, \qquad t = 0, \pm 1, \pm 2, \ldots$$

Having found p_0, q_0, then cp_0, cq_0 will be a general solution to $aq - bp = c$ and all other solutions will be of the form

$$cp_0 + at$$
$$cq_0 + bt, \qquad t = 0, \pm 1, \pm 2, \ldots$$

The smallest positive solution will be when $-t$ is the integer part of the smaller of cp_0 / a and cq_0 / b.

Applying this to the equation at hand, we find that the particular solution of $aq - bp = 1$ is $(p_0, q_0) = (1097, 501)$. Then $cp_0 = 7{,}049{,}322$ and so the smallest positive solution for p in $aq - bp = c$ is 9,322 so we now know that

$$\begin{aligned} R &= \rho + p \\ &= 146{,}440{,}678 + 9{,}322 \\ &= 144{,}450{,}000 \end{aligned}$$

is a number of revolutions that will give a mean longitude of zero after 1,972,944,000 years.

Our final step is to adjust the trailing digits of $R = 146{,}450{,}000$ to give a desired mean longitude at our 'modern' date, 3,600 years after –3101 Feb 18. The clue we need to resolve this is already implicit in the analysis given above, where it was shown that the solution of

$$10{,}000q - 4567p = 1.$$

is $(p_0, q_0) = (1097, 501)$, or

$$0.4567 \times 1097 = 501 - \frac{1}{10,000}.$$

This means that if we add the final four digits of a small multiple of 1097 to our provisional R = 146,560,000 we can keep the mean longitude at –3103 Feb 18 close to zero (we lower it by 1/10,000 revolutions, or 0;2,9°, for each multiple of 1097). In fact, we see that

$$360^\circ \times \frac{3,600}{4,320,000,000} \times 1097 = 0;19,44^\circ$$

and so the net mean longitude after 146,561,097 revolutions over 1,972,944,000 + 3,600 years is 0;19,44° – 0;2,9° = 0;17,35° and each additional multiple of 1097 revolutions that we add increases the net mean longitude by that same amount.

The numbers actually used for each planet in the *Paita* and the *BSS* are

Saturn	34×1097	yielding	146,56*7,298*	and	$\bar{\lambda}$ = 48;58° (49;12°),
Jupiter	15×1097	yielding	364,*226,455*	and	$\bar{\lambda}$ = 187;24° (187;12°),
Mars	26×1097	yielding	2,296,82*8,522*	and	$\bar{\lambda}$ = 7;37° (7;12°),
Venus	36×1097	yielding	7,022,38*9,492*	and	$\bar{\lambda}$ = 355;33° (356;24°),
Mercury	72×1097	yielding	17,*9*36,99*8,984*	and	$\bar{\lambda}$ = 177;06° (180°),
Moon	0×1097	yielding	57,753,30*0,000*	and	$\bar{\lambda}$ = 270;00° (280;48°),

A similar analysis explains the revolution numbers for the nodes and apogees

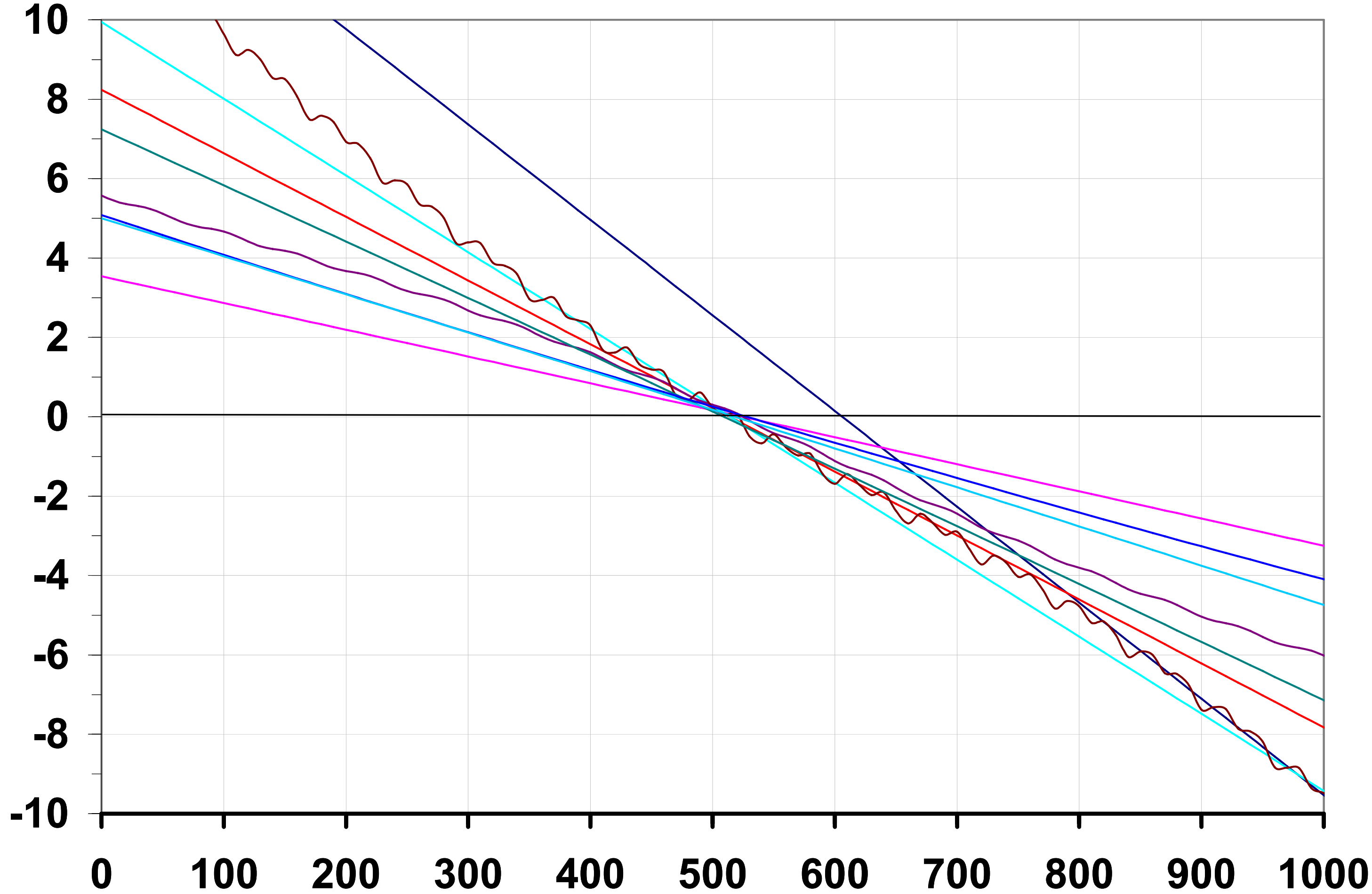
apogees	revolutions
Mercury	332
Venus	653
Sun	480
Mars	292
Jupiter	855
Saturn	41
nodes	revolutions
Mercury	-521
Venus	-893
Mars	-267
Jupiter	-63
Saturn	-584

Source of the Mean Longitudes

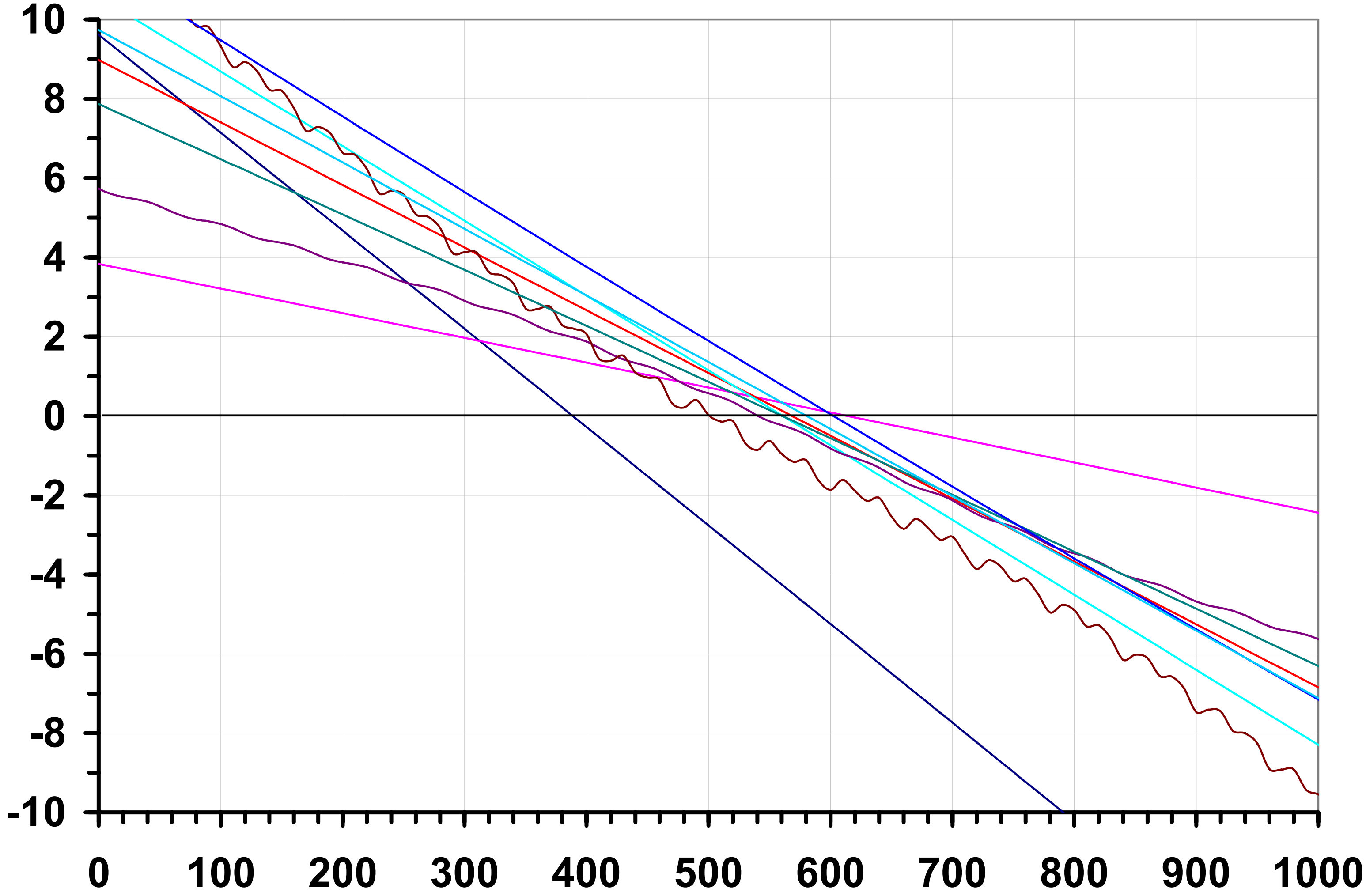
There are two extreme possibilities: either the mean longitudes were taken from some available astronomical tables, e.g. those in the *Almagest* or the *Handy Tables*, or they were derived from observations made in India around A.D. 500. The principal advocate for the case that tables were used was Pingree, while the principal advocates for observation in India were Billard and van der Waerden.

Billard's thesis is that by plotting the errors as a function of time, one could determine the date of the observations by finding when the errors, or more precisely their standard deviation, were minimized.

Billard's error plot for the Aryabhata sunrise system



the same plot for *Paita* and the *Brahmasphutasiddhanta*



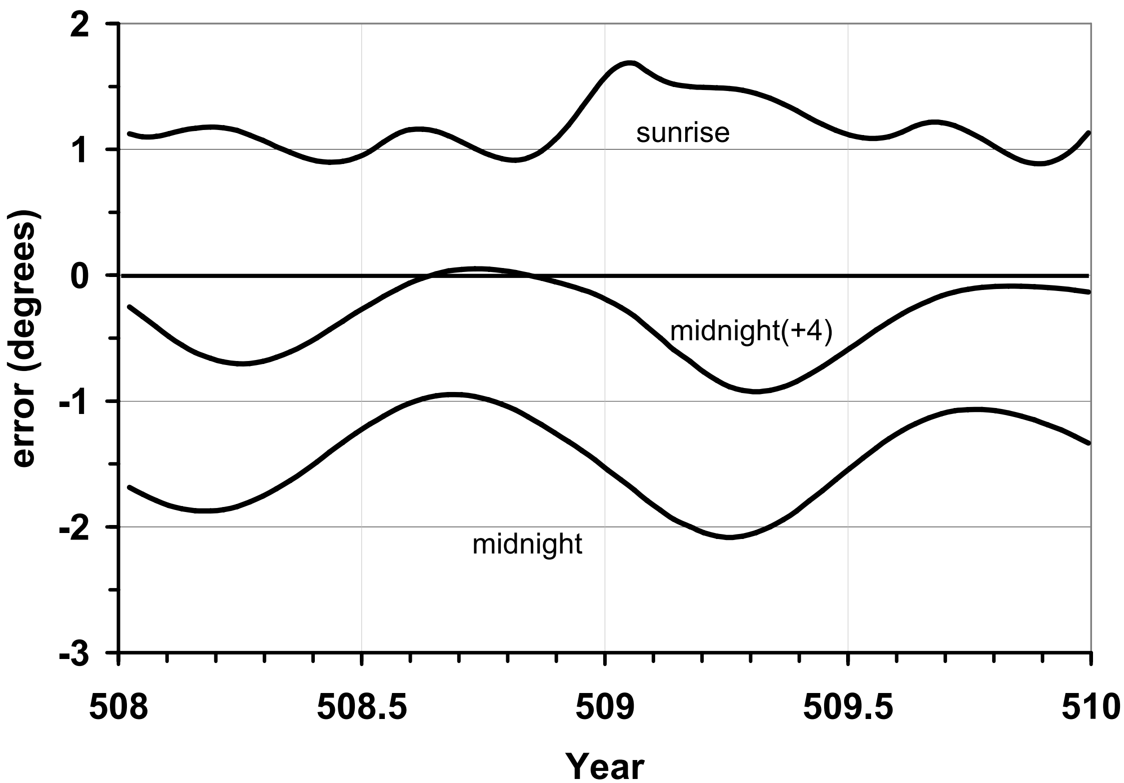
Generally speaking, mean longitudes are not directly observable. Rather it is necessary to observe true longitudes, and from these, using a model with known structural parameters such as eccentricity, epicycle radius, and longitude of the apogee, one can compute the desired mean longitudes. In the case of the Indian astronomers, there are three main problems:

(1) there is no evidence that the astronomers would be able to measure true longitudes with any accuracy. The surviving lists of star coordinates in the Indian texts are very imprecise and so no measurement of a planet relative to a star would be accurate.

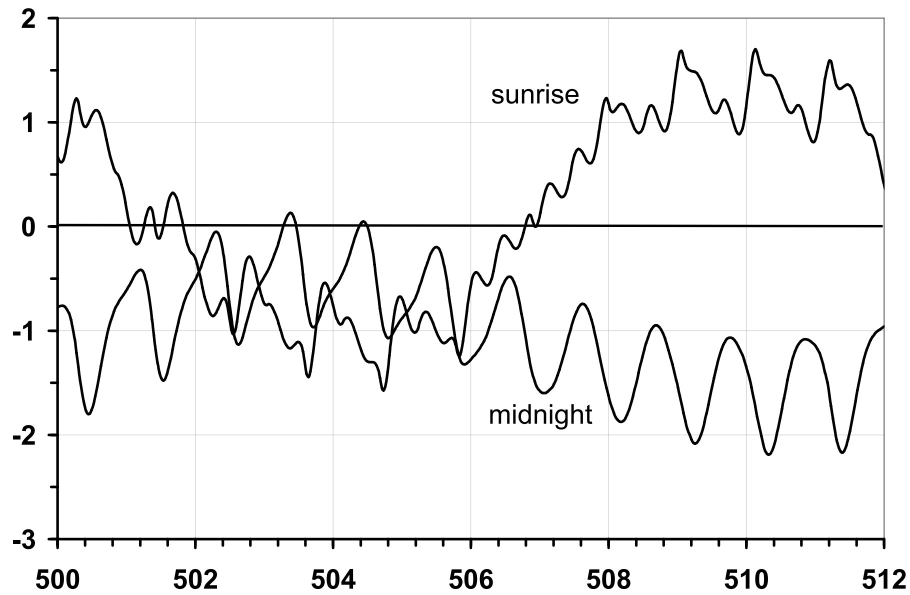
(2) the structural parameters given in the texts are often not at all accurate, especially for the planets.

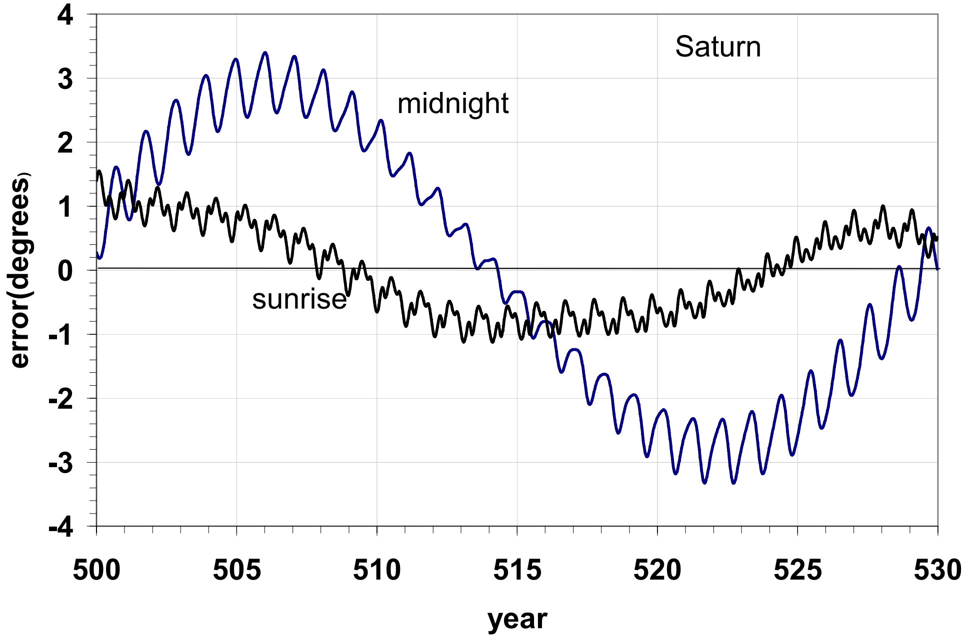
(3) the kinematic models used by the Indian astronomers are at best approximations to good models such as the equant, and for Mercury, Venus, and Mars, not very good approximations at that.

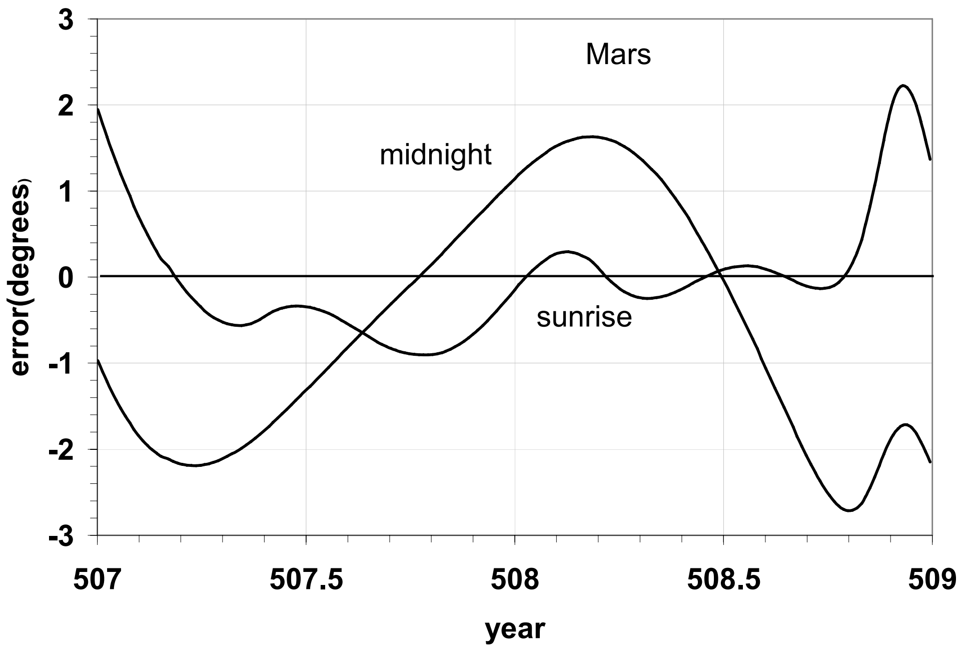
These problems make determination of an accurate mean longitude from a set of observed true longitudes very difficult. Let us illustrate these difficulties with the case of Jupiter. In the midnight system we have R = 364,220 and the mean longitude is 186°.

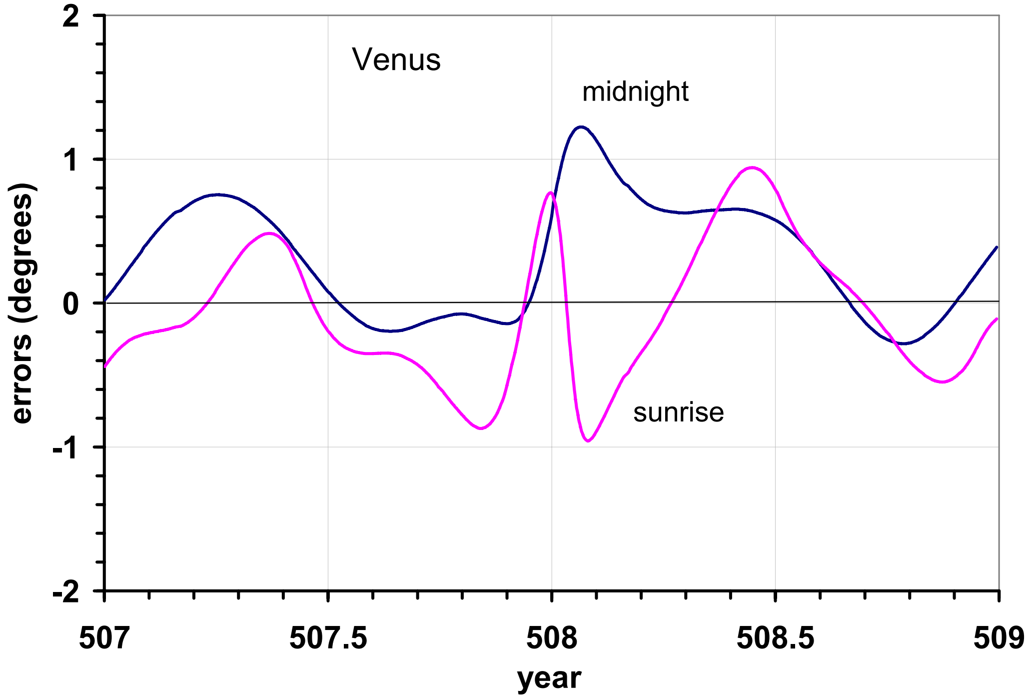


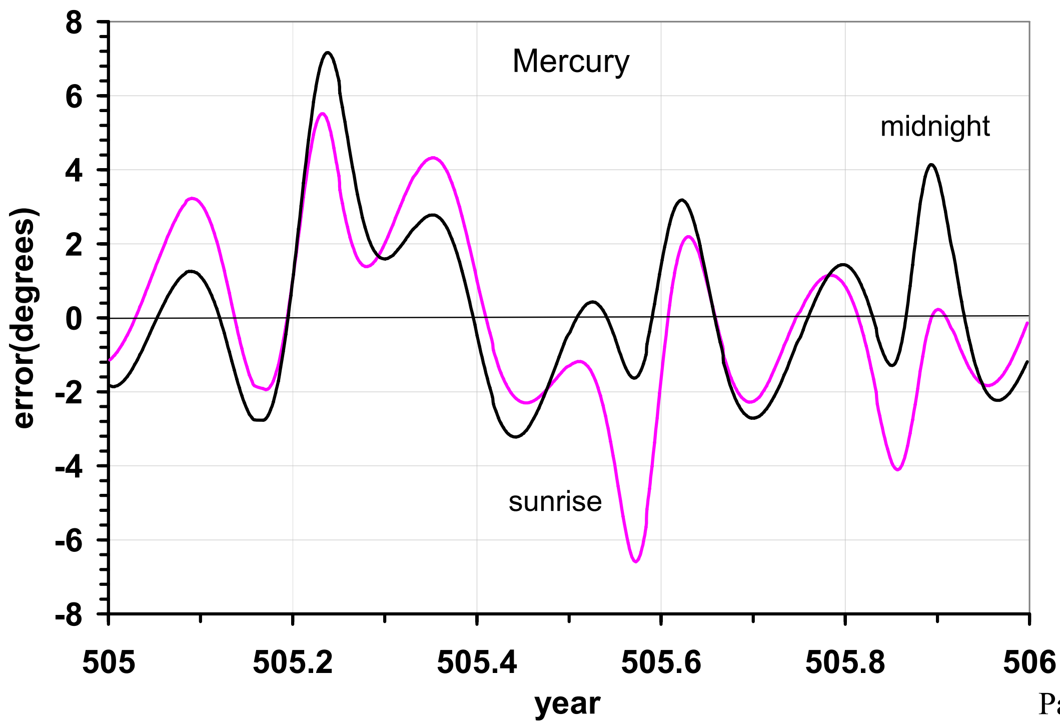
However, this is not all our astronomer actually changed in going to the sunrise system. Small adjustments are made to the eccentricity and the epicycle radius and small pulsations are added, and on the scale of the inherent uncertainties in the entire process, all these changes are inconsequential. What is significant is that the apogee is moved from 160° to 180°. There is clearly no evidence of accurate observations being turned into excellent agreement with the sunrise or midnight systems.




Saturn
midnight
sunrise
error(degrees)
year
4
3
2
1
0
-1
-2
-3
-4
500
505
510
515
520
525
530


Mars
midnight
sunrise
error(degrees)
year
3
2
1
0
-1
-2
-3
507
507.5
508
508.5
509


Venus
midnight
sunrise
errors (degrees)
2
1
0
-1
-2
507
507.5
508
508.5
509
year


Mercury
midnight
sunrise
error(degrees)
8
6
4
2
0
-2
-4
-6
-8
505
505.2
505.4
505.6
505.8
506
year