UNIFORMITY MEASURES FOR POINT SAMPLES IN HYPERCUBES

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Uniformly distributed point samples have three attributes:

- the points are equally spaced;
- the points cover the region, i.e., there do not exist relatively large subregions that contain no points; and
- the points are isotropically distributed, i.e., there is no directional bias in the placement of points.

In 2D, the eye does a great job in using all three attributes to compare the uniformity of different point sets. Some popular quantative measures of uniformity are flawed in that they only consider the spacing between points.

The measures discussed here make no attempt to look at the uniformity of projections of point sets onto lower dimensional faces of the hypercube. In particular, the measures give no information about the discrepancy of the point sets. Thus, these measures should be viewed as being useful for determining the uniformity of point sets in the hypercube itself; the relative merits of different point sets as determined by the measures discussed here apply to applications for which volumetric uniformity is most important.

We look at two types of uniformity measures. Point-to-point uniformity measures take into account the distances between pairs of points; these are good for determining how uniformly the points are spaced. Volumetric uniformity measures attempt to measure one or both of the other two attributes.

We present eight uniformity measures and their application to several point sets in 2D and 7D hypercubes. In the appendices, we give some details about the last two measure we present.

POINT-TO-POINT UNIFORMITY MEASURES

1. COV measure λ

Given any set of N points $\{\mathbf{z}_i\}_{i=1}^N$, we compute

$$\gamma_i = \min_{j \neq i} |\mathbf{z}_i - \mathbf{z}_j|$$

so that γ_i is the minimum distance between the point $\mathbf{z}_i$ and any of the other points. Then, the *COV measure* λ is given by

$$\lambda = \frac{1}{\overline{\gamma}} \left(\frac{1}{N} \sum_{i=1}^{N} (\gamma_i - \overline{\gamma})^2 \right)^{1/2} = \left(-1 + \frac{1}{\overline{\gamma}^2 N} \sum_{i=1}^{N} \gamma_i^2 \right)^{1/2} = \left(N \frac{\sum_{i=1}^N \gamma_i^2}{\left(\sum_{i=1}^N \gamma_i \right)^2} - 1 \right)^{1/2}$$

where

$$\overline{\gamma} = \frac{1}{N} \sum_{i=1}^{N} \gamma_i$$

For a perfectly uniform mesh, $\gamma_1 = \gamma_2 = \cdots = \gamma_N = \overline{\gamma}$ so that $\lambda = 0$. Thus, the smaller λ is, the more uniform is the mesh.

2. The mesh ratio γ

Given any set of N points $\{\mathbf{z}_i\}_{i=1}^N$, the *mesh ratio* γ is given by

$$\gamma = \frac{\max_{i=1,\dots,N} \gamma_i}{\min_{i=1,\dots,N} \gamma_i}$$

For a perfectly uniform mesh, $\gamma_1 = \gamma_2 = \cdots = \gamma_N$ so that $\gamma = 1$. The smaller γ is, i.e., the closer it is to unity, the more uniform is the mesh.

VOLUMETRIC UNIFORMITY MEASURES

Given any set of N points $\{\mathbf{z}_i\}_{i=1}^N$ in a region Ω, we can use those points to generate a Voronoi tessellation $\{V_i\}_{i=1}^N$ of Ω. Then, we can associate with each point a corresponding Voronoi region and then determine various quantities associated with the points and the regions that can be used to measure the quality of the set of points

3. The point distribution norm h

Given a Voronoi tessellation $\mathcal{V} = \{\mathbf{z}_i, V_i\}_{i=1}^N$, the *point distribution norm* h is given by

$$h = \max_{i=1,\dots,N} h, \qquad \text{where} \qquad h_i = \max_{\mathbf{y}\in V_i} |\mathbf{z}_i - \mathbf{y}|$$

Thus, h_i gives the maximum distance between the particular generator $\mathbf{z}_i$ and the points in its associated cell V_i and h gives the maximum distance between any generator and the points in its associated cell

The point distribution norm h can be used as a measure of the uniformity of point distribution, i.e., of how "close" a point distribution is to an ideal uniform point distribution; the smaller the value of h, the more uniform is the point distribution

4. The point distribution ratio μ

Given a Voronoi tessellation $\mathcal{V} = \{\mathbf{z}_i, V_i\}_{i=1}^N$, the *point distribution ratio* μ is given by

$$\mu = \frac{\max_{i=1,\dots,N} h_i}{\min_{i=1,\dots,N} h_i}, \qquad \text{where} \qquad h_i = \max_{\mathbf{y}\in V_i} |\mathbf{z}_i - \mathbf{y}|$$

For an ideal uniform point distribution, $\mu = 1$ so that the smaller μ is, i.e., the closer it is to unity, the more uniform the point distribution.

5. The regularity measure χ

Given a Voronoi mesh $\mathcal{V} = \{\mathbf{z}_i, V_i\}_{i=1}^N$, we define the *regularity measure* χ by

$$\chi = \max_{i=1,\dots,N} \chi_i, \qquad \text{where} \qquad \chi_i = \frac{2h_i}{\gamma_i}$$

For an ideal uniform mesh, $\chi = \chi_i$ for all i; any deviation from uniformity will increase the value of χ. Thus, χ can be used as another measure of the uniformity of a mesh; the smaller the value of χ, the more uniform is the mesh. In addition, the value of χ provides us a measure of the mesh regularity, i.e., the *local uniformity* of a mesh. Again, if a mesh is locally uniform in the sense that the cells in a neighborhood of any cell are nearly congruent to that cell, then the value of χ will again be small

6. Cell volume deviation ν

Given a Voronoi mesh $\mathcal{V} = \{\mathbf{z}_i, V_i\}_{i=1}^N$, we define the *cell volume deviation* ν by

$$\nu = \frac{\max_{i=1,\dots,N} |V_i|}{\min_{i=1,\dots,N} |V_i|}$$

where V_i denotes the volume of the cell V_i

For a perfectly uniform distribution of N points $\{\mathbf{z}_i\}_{i=1}^N$ in a given region Ω, the corresponding volumes $|V_i|$ would all be equal, i.e., $|V|_1 = |V|_2 = \cdots = |V|_N$ so that $\nu = 1$. Thus, the smaller is ν, i.e., the closer it is to unity, the better is the uniformity of the point distribution.

7. The second moment trace measure τ

Given a Voronoi mesh $\mathcal{V} = \{\mathbf{z}_i, V_i\}_{i=1}^N$, let T_i denote the trace of the second moment tensor (about the region generator) associated with each Voronoi region V_i. Let $\overline{T} = \frac{1}{N}\sum_{i=1}^N T_i$ denote the average of the trace over the n regions. Then, we define the *second moment trace measure* τ by

$$\tau = \max_{i=1,\dots,n} |T_i - \overline{T}| \,.$$

For a perfectly uniform distribution of N points $\{\mathbf{z}_i\}_{i=1}^N$ in a given region V, we would have $T_1 = T_2 = \cdots = T_N = \overline{T}$ so that $\tau = 0$. Thus, the smaller τ is the better is the uniformity of the point distribution.

8. The second moment determinant measure d

Given a Voronoi mesh $\mathcal{V} = \{\mathbf{z}_i, V_i\}_{i=1}^N$, let D_i denote the determinant of the deviatoric tensor associated with each Voronoi region V_i. Then, the *second moment determinant measure* d

$$d = \max_{i=1,\dots,n} |D_i| \,.$$

For a perfectly uniform distribution of N points $\{\mathbf{z}_i\}_{i=1}^N$ in a given region V, we would have $D_1 = D_2 = \cdots = D_N = 0$ so that $\tau = 0$. Thus, the smaller τ is the better is the uniformity of the point distribution.

VISUAL COMPARISONS OF POINT SETS IN 2D

The figures display 3 Monte Carlo points sets, a Halton point set, a Hammersley point set, and 3 LHS point sets in the square, each set having 100 points. Below each figure is the CVT point set obtained by using the upper point set as an initial condition for the CVT iteration.

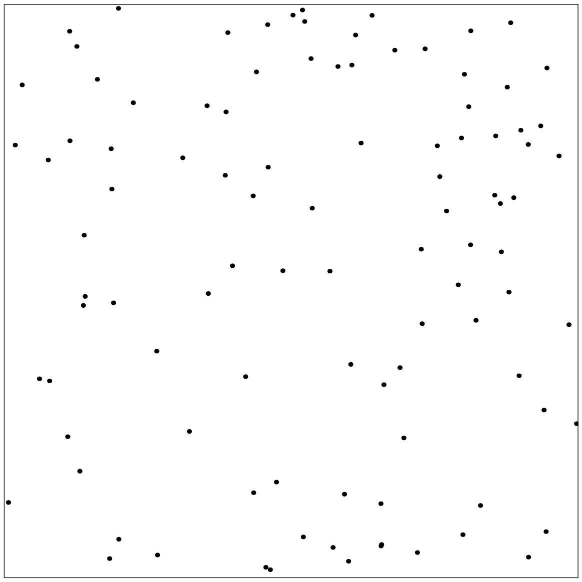

Figure 1: Monte Carlo point set 1

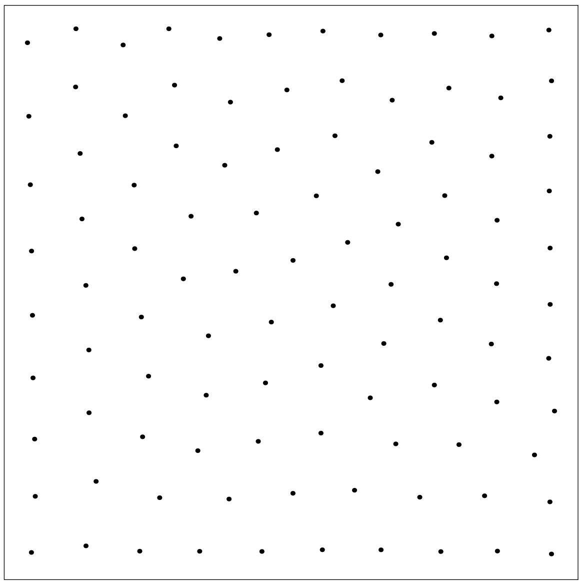

Figure 2: CVT point set obtained from Monte Carlo point set 1

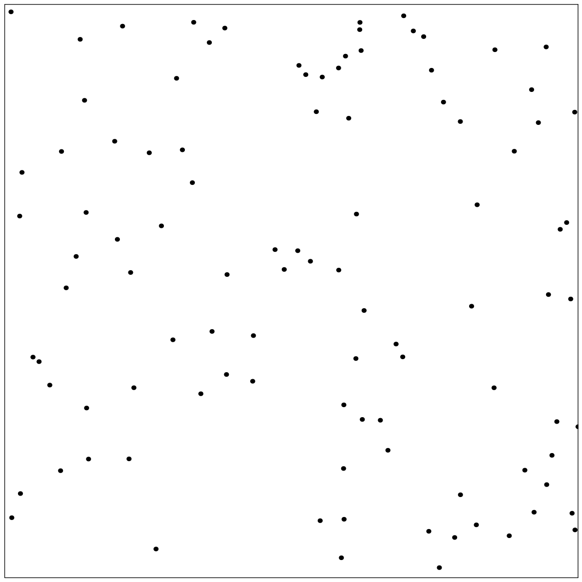

Figure 3: Monte Carlo point set 2

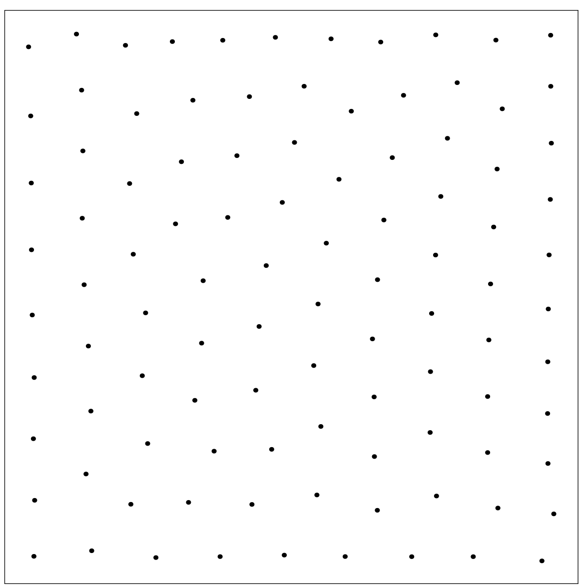

Figure 4: CVT point set obtained from Monte Carlo point set 2

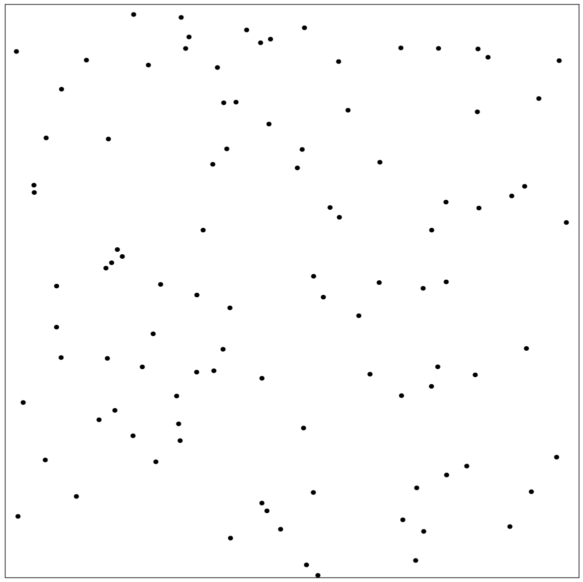

Figure 5: Monte Carlo point set 3

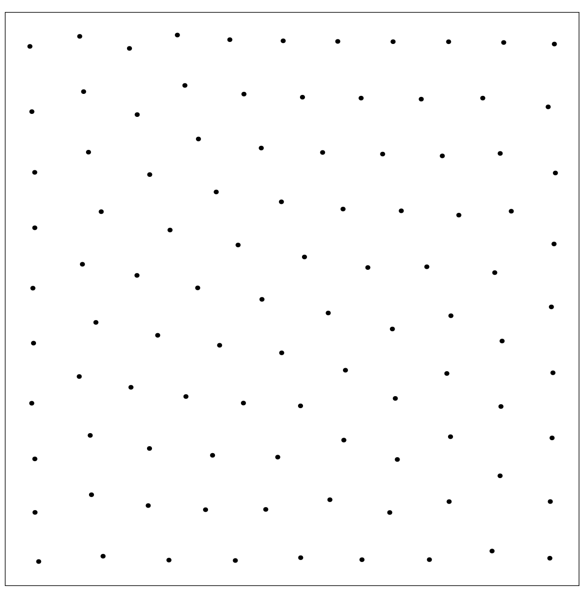

Figure 6: CVT point set obtained from Monte Carlo point set 3

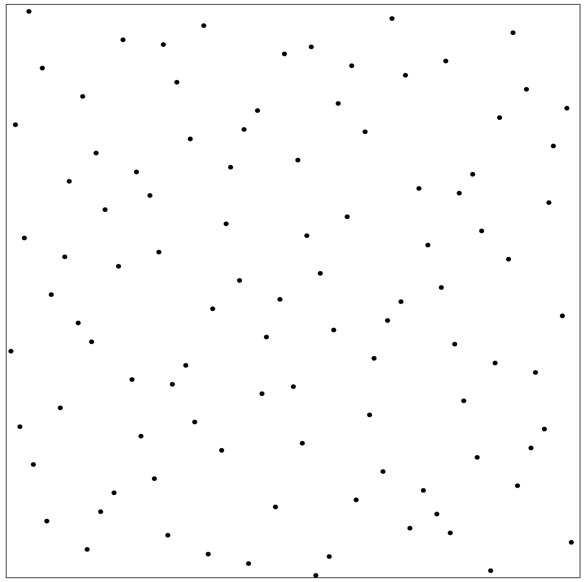

Figure 7: Halton point set

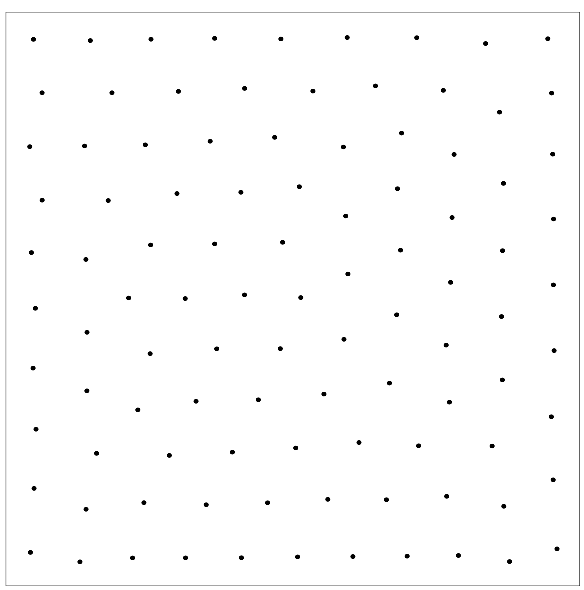

Figure 8: CVT point set obtained from Halton point set

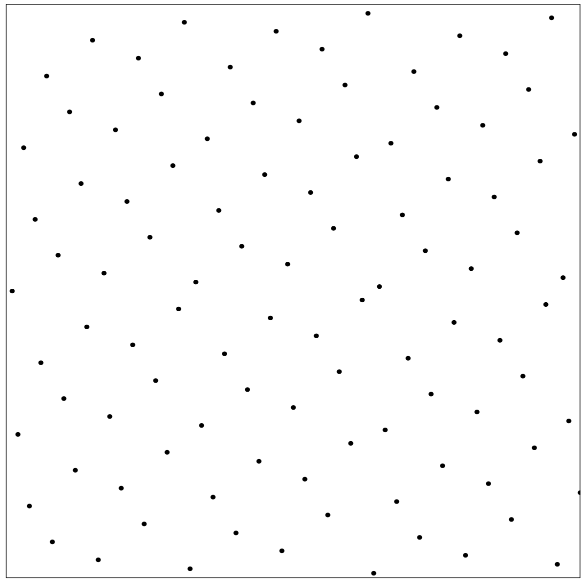

Figure 9: Hammersley point set

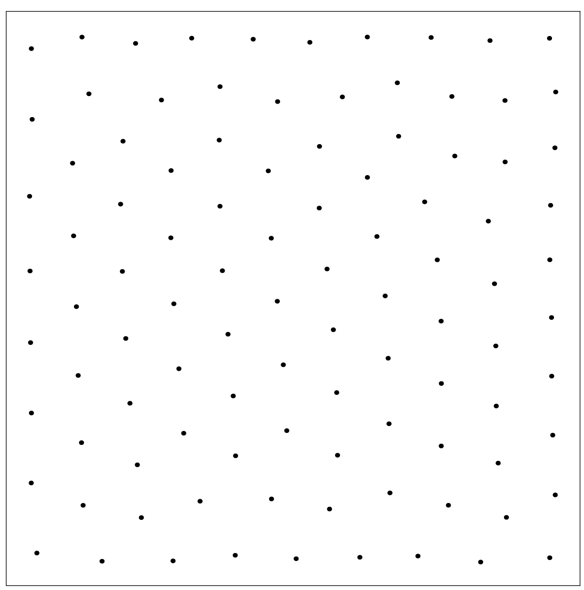

Figure 10: CVT point set obtained from Hammersley point set

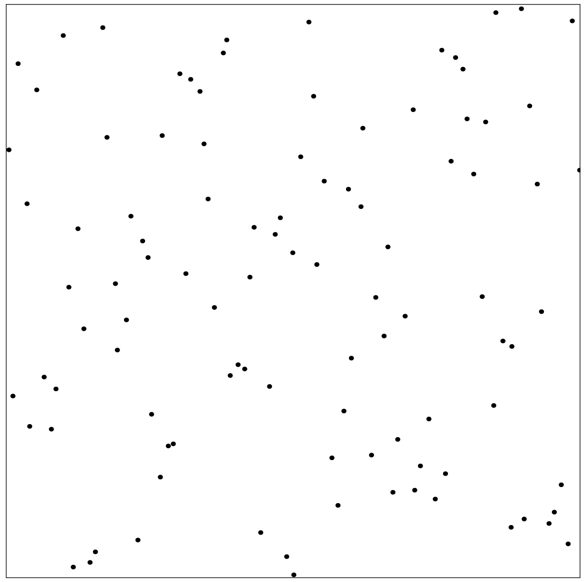

Figure 11: LHS point set 1

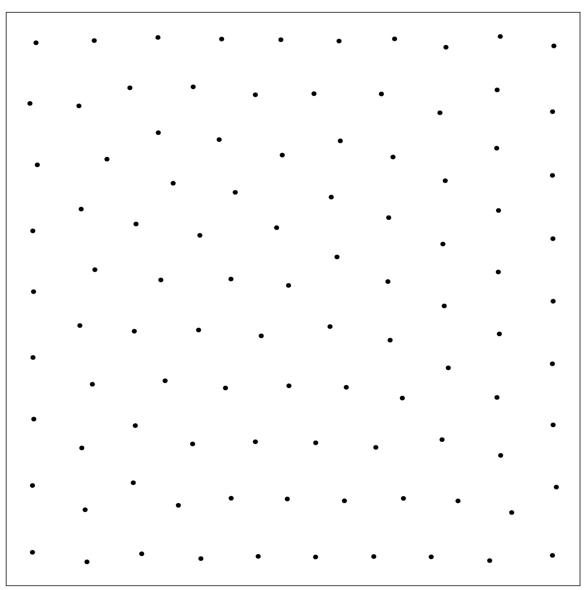

Figure 12: CVT point set obtained from LHS point set 1

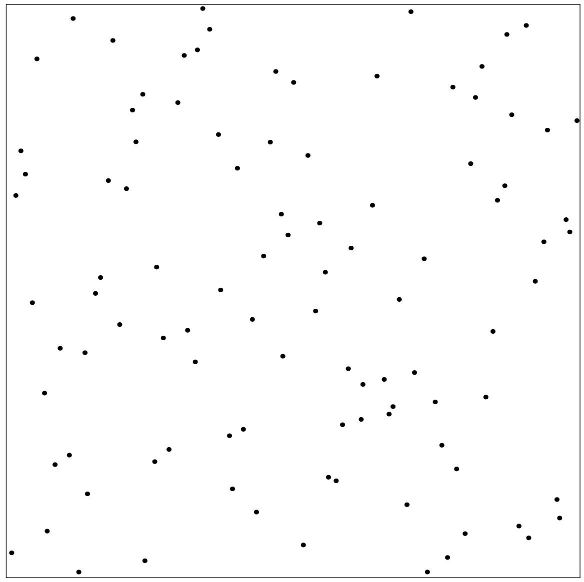

Figure 13: LHS point set 2

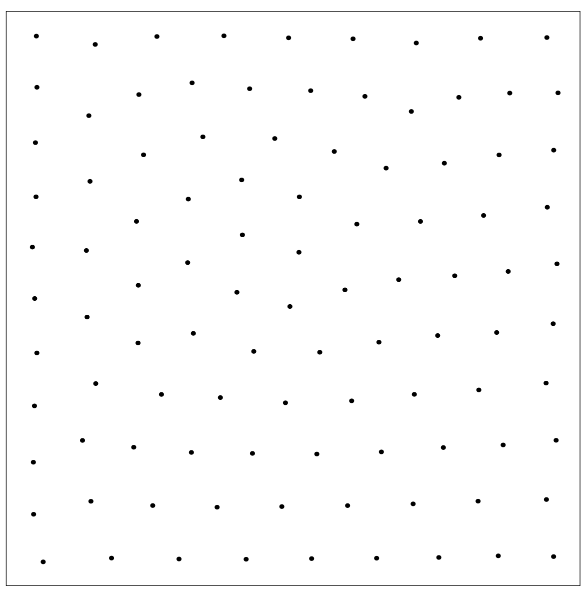

Figure 14: CVT point set obtained from LHS point set 2

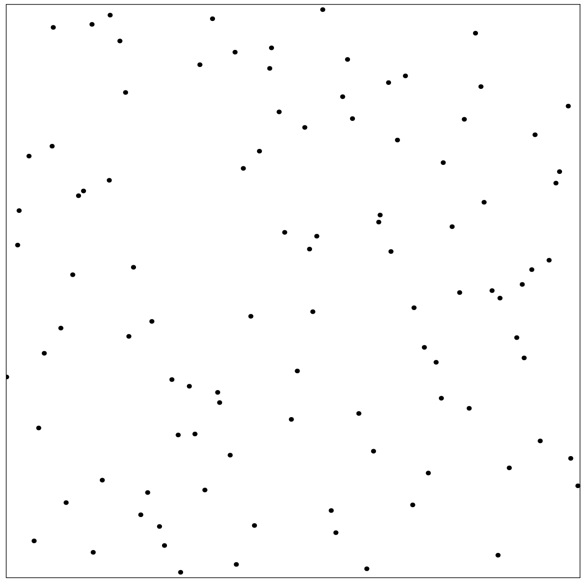

Figure 15: LHS point set 3

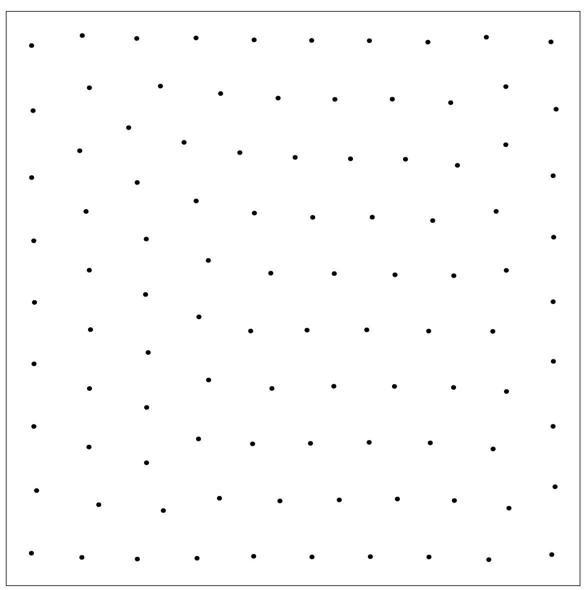

Figure 16: CVT point set obtained from LHS point set 3

QUANTITATIVE COMPARISONS OF POINT SETS IN 2D AND 7D

The plots in the next four pages show the 8 uniformity measures for each of the point samples shown on the previous 8 pages. Each plot is for a different uniformity measure and shows the value of that measure for the four point sets determined by the Monte Carlo, Halton, Hammersley, and LHS sampling methods and the CVT points determined starting from these.

The tables following the plots also give the values of the 8 uniformity measures for several sampling methods. In addition to Monte Carlo, Halton, Hammersley, LHS, and CVT, we include and improved LHS method (labeled IHS) as well as three additional quasi-Monte Carlo sampling methods. For probabilistically determined points samples, e.g., Monte Carlo, the values given are the averages over several realizations.

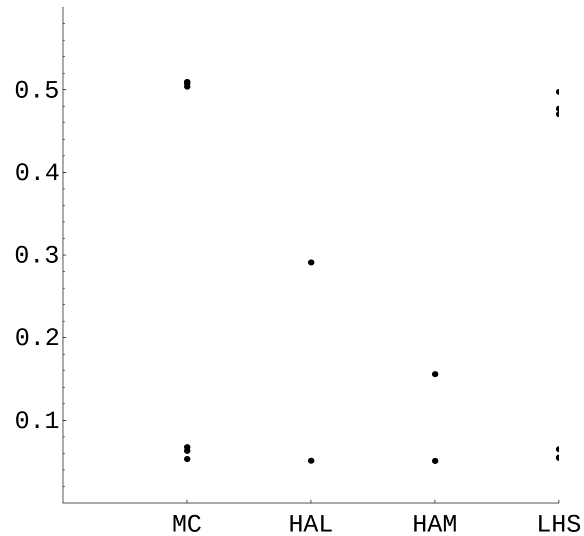

Figure 1: COV quality measure. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

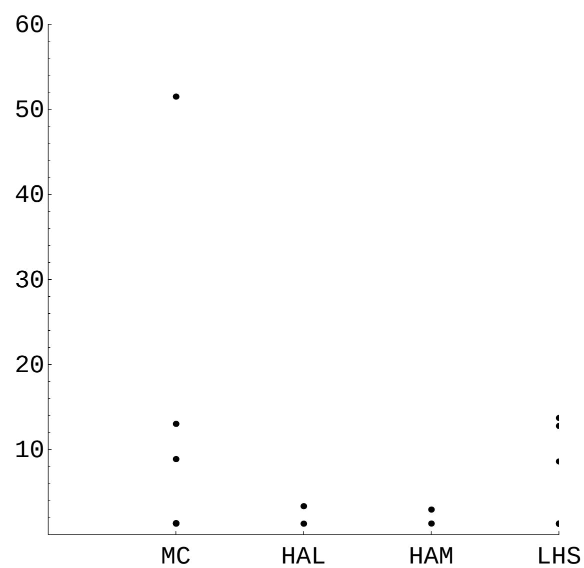

Figure 2: Mesh ratio quality measure γ. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

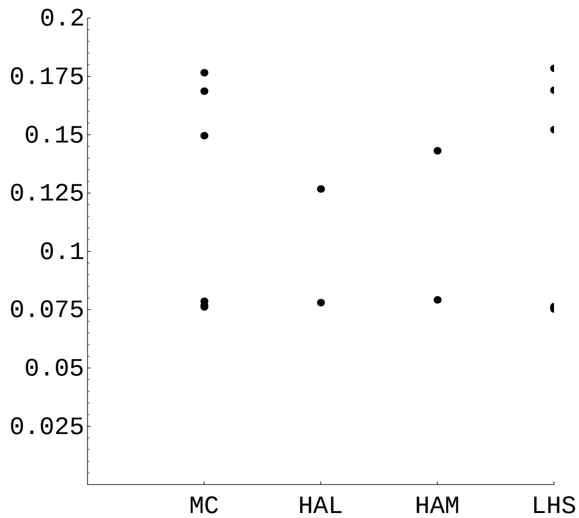

Figure 3: Mesh norm quality measure h. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

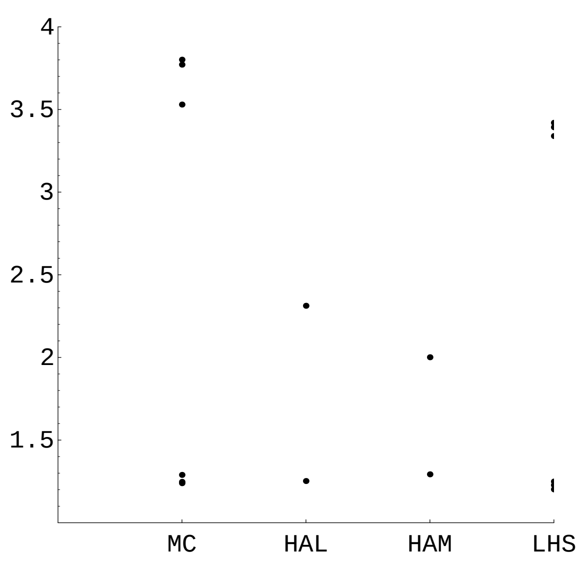

Figure 4: Point distribution ratio quality measure μ. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

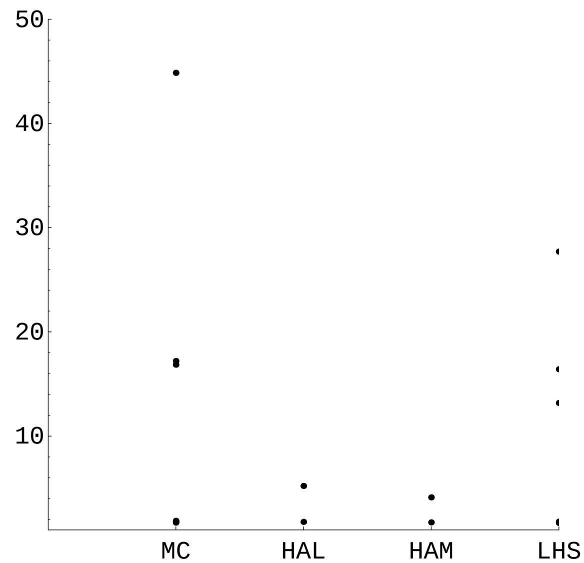

Figure 5: Regularity quality measure χ. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

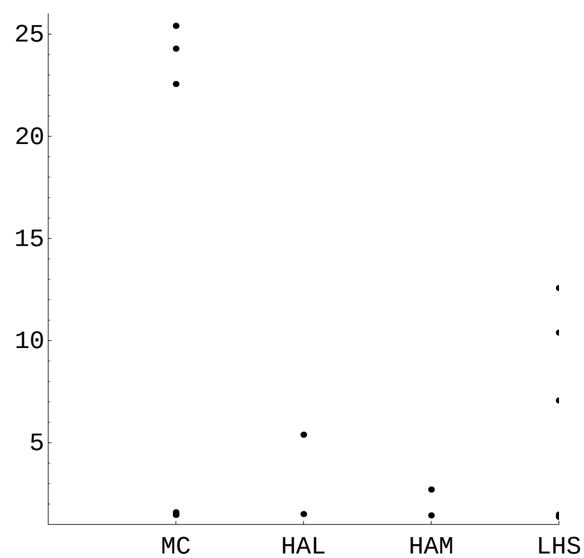

Figure 6: Cell volume quality measure ν. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

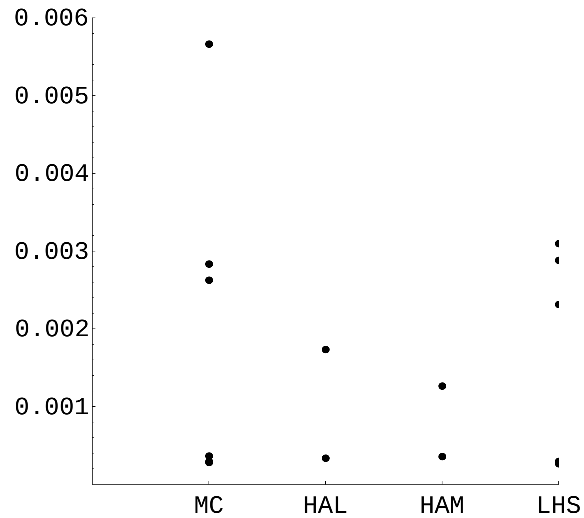

Figure 7: 2nd moment trace quality measure τ. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

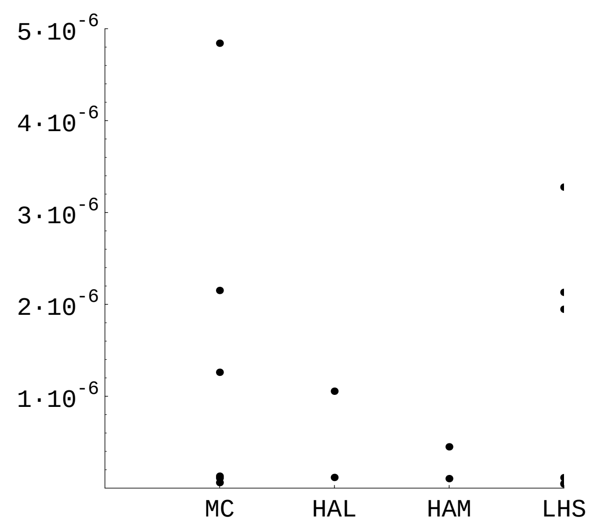

Figure 8: 2nd moment determinant quality measure d. Upper points are for labeled type of point set. Lower points are for CVT points generated from corresponding point sets.

	COV	γ	h	μ	χ	ν	τ	d
Ideal	0	1	0.0707	1	1.414	1	0	0
Monte Carlo	0.5075	88.75	0.1767	3.792	17.17	25.11	0.2833	0.01246
Halton	0.2911	3.37	0.1266	2.325	5.213	5.386	0.1732	0.01057
Hammersley	0.1559	2.94	0.1424	2.003	4.084	2.744	0.1257	0.00457
Faure	0.2552	2.70	0.1472	2.245	4.640	3.081	0.1432	0.00926
Sobol	0.5246	12.55	0.1378	2.068	20.16	3.369	0.1453	0.01865
Niederreiter	0.3072	3.05	0.1279	1.879	5.172	3.026	0.1294	0.01630
Latin hypercube	0.4771	8.60	0.1690	3.382	13.11	10.04	0.2907	0.02130
IHS	0.1588	2.46	0.1225	2.115	5.551	3.103	0.1033	0.00549
CVT	0.0509	1.30	0.0792	1.311	1.720	1.456	0.0355	0.00107

Eight measures of uniformity for different types of 100 “uniformly” distributed point in the unit square

Red – flawed measures of sample uniformity

Green – good measures of sample uniformity

	h	χ	τ	d
Monte Carlo	0.100874D+01	0.608636D+01	0.520662D-01	0.195518D-12
Halton	0.923242D+00	0.458080D+01	0.295432D-01	0.686130D-13
Hammersley	0.920083D+00	0.481531D+01	0.277011D-01	0.441584D-13
Latin hypercube	0.900931D+00	0.652482D+01	0.279853D-01	0.773161D-13
CVT	0.719571D+00	0.277510D+01	0.725527D-02	0.194127D-17

Four "good" measures of uniformity for different types of 100 "uniformly" distributed point in the 7D square

APPENDIX I. MOMENTS OF UNIFORM POINT DISTRIBUTIONS

Given a set of points $\{\mathbf{z}_i\}_{i=1}^N$ in a region V in k-dimensional Euclidean space, we associate with each point $\mathbf{z}_i$ the Voronoi cell consisting of all points in V that are closer to $\mathbf{z}_i$ than to any other point $\mathbf{z}_j$, $j \neq i$.

For $i = 1, \ldots, N$, the *zeroth moment* or *volume* of a region V_i (a scalar) is defined by

$$|V|_i = \int_{V_i} d\mathbf{x}\,,$$

the *first moment* or the *center of mass* or the *cetnroeid* of V_i (a vector) is defined by

$$\overline{\mathbf{x}}_i = \frac{1}{|V|_i} \int_{V_i} \mathbf{x}\, d\mathbf{x}\,,$$

and the *second moments* (relative to the center of mass) of V_i (a tensor) by

$$\mathbb{M}_i = \frac{1}{|V|_i} \int_{V_i} (\mathbf{x} - \overline{\mathbf{x}}_i)(\mathbf{x} - \overline{\mathbf{x}}_i)^T\, d\mathbf{x}\,.$$

Since we have that

$$\mathbb{M}_i = \mathbb{M}_{0i} - \overline{\mathbf{x}}_i \overline{\mathbf{x}}_i^T\,,$$

where $\mathbb{M}_{0i}$ denotes the second-order moment tensor relative to the origin, i.e.,

$$\mathbb{M}_{0i} = \frac{1}{|V|_i} \int_{V_i} \mathbf{x}\mathbf{x}^T\, d\mathbf{x}\,,$$

it is clear that if, for each V_i, one determines the three integrals

$$|V|_i = \int_{V_i} d\mathbf{x}\,, \qquad \mathbf{q}_i = \int_{V_i} \mathbf{x}\, d\mathbf{x}\,, \qquad \text{and} \qquad \mathbb{S}_i = \int_{V_i} \mathbf{x}\mathbf{x}^T\, d\mathbf{x}\,,$$

then one can easily compute the desired quantitiess. In fact, we simply have that

$$\overline{\mathbf{x}}_i = \frac{1}{|V|_i}\mathbf{q}_i \qquad \text{and} \qquad \mathbb{M}_i = \frac{1}{|V|_i}\mathbb{S}_i - \overline{\mathbf{x}}_i \overline{\mathbf{x}}_i^T\,.$$

APPENDIX II. MOMENTS OF UNIFORM POINT DISTRIBUTIONS AS UNIFORMITY AND COVERAGE MEASURES

Given any set of points in a region V, we can use those points to generate a Voronoi tessellation of V. Then, we can associate with each point a corresponding Voronoi region and then determine various quantities associated with the points and the regions, including the zeroth, first, and second-order moments for the regions. These can then be used to determine the quality of the set of points.

We will use three particular properties associated with the set of points and the corresponding Voronoi regions as measures of the quality, i.e., the uniformity, of a point set. These are:

- the zeroth-order moments $|V_i|$ (the volumes) of the Voronoi regions associated with each point;
- the trace $T_i = \text{trace}(\mathbb{M}_i)$ of the second-moment matrix associated with each point and its corresponding Voronoi region[1]; and
- the determinant D_i of the deviatoric matrix $\mathbb{M}_i - \overline{M}_i \mathbb{I}$ associated with each point and its corresponding Voronoi region[2] , where $\mathbb{I}$ denotes the identity matrix and where $\overline{M}_i = T_i/k$.

For a *perfectly uniform distribution* of N points $\{\mathbf{z}_i\}_{i=1}^N$ in a given region V, the corresponding volumes $|V_i|$ would all be equal, i.e., $|V_1| = |V_2| = \cdots = |V_N|$, the corresponding traces $\{T_i\}_{i=1}^N$ would also all be equal, i.e., $T_1 = T_2 = \cdots = T_N$, and the corresponding determinants $\{D_i\}_{i=1}^N$ would all vanish, i.e., $D_1 = D_2 = \cdots = D_N = 0$. Thus, we can use the uniformity of these volumes and traces and the smallness of these determinants as measures of the uniformity of a set of points.

[1]The trace of $\mathbb{M}_i$ is given by

$$T = \sum_{j=1}^{k} M_{jj}^{(i)}$$

where $M_{jk}^{(i)}$ denotes the j, k element of $\mathbb{M}_i$.

[2]For each point $\mathbf{z}_i$, the determinant D_i of the corresponding deviatoric or second-moment matrix $\mathbb{M}_i - \overline{M}_i \mathbb{I}$, is given by, for example,

$$D_i = \det \begin{pmatrix} M_{11}^{(i)} - \overline{M}_i & M_{12}^{(i)} \\ M_{12^{(i)}} & M_{22}^{(i)} - \overline{M}_i \end{pmatrix} \qquad \text{in two dimensions}$$

$$D_i = \det \begin{pmatrix} M_{11}^{(i)} - \overline{M}_i & M_{12}^{(i)} & M_{13}^{(i)} \\ M_{12}^{(i)} & M_{22}^{(i)} - \overline{M}_i & M_{23}^{(i)} \\ M_{13}^{(i)} & M_{23}^{(i)} & M_{33}^{(i)} - \overline{M}_i \end{pmatrix} \qquad \text{in three dimensions.}$$