This is a Fine Mesh You've Gotten Me Into!

John Burkardt

16 November 2021, 11am, 427 Thackeray, Undergraduate Math Seminar, Jeff Wheeler, moderator.

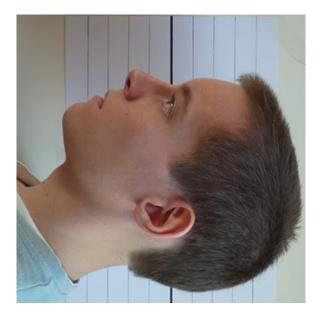
University of Pittsburgh Mathematics Department

 $\label{eq:https://people.sc.fsu.edu/~jburkardt/presentations/...} \\ \hfill \dots mesh_2021_pitt.pdf$

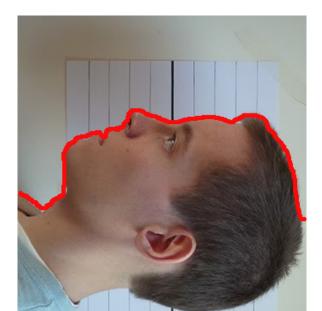
2: A Profile (Not Mine!)



3: A Profile Rotated



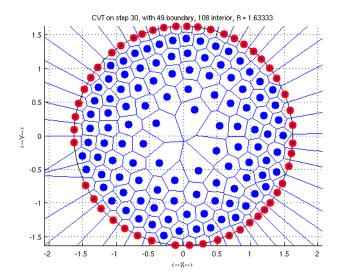
4: A Profile Rotated and Outlined



5: A Profile Rotated and Outlined and Extracted

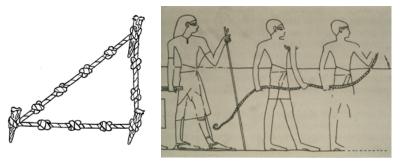


6: A Flower Grown In a Computer?



https://people.sc.fsu.edu/~jburkardt/m_src/cvt_corn/step30.png

7: Meshes Started with Taxation

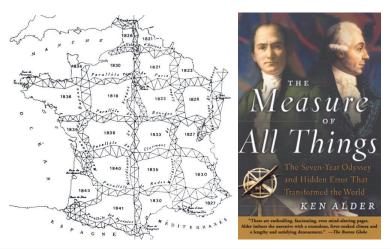


8: A Mesh Organizes and Simplifies a Complicated View



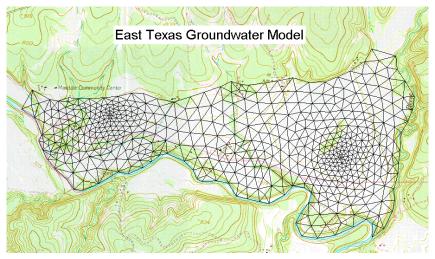
https://www.metmuseum.org/art/collection/search/366555

9: A Mesh Can Organize and Measure Curved Surfaces



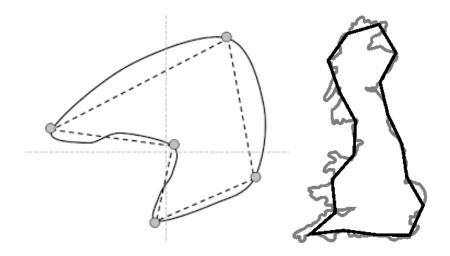
https://www.simonandschuster.com/books/The-Measure-of-All-Things/Ken-Alder/9780743216760

10: Computers Model Reality with Meshes

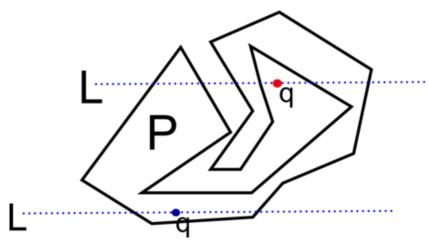


https://www.caee.utexas.edu/prof/maidment/gishyd97/gms/gms.htm

11: Computers Approximate Shapes with Polygons



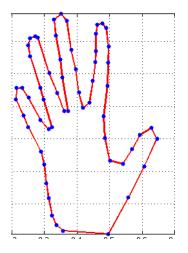
12: Arithmetic is Easy, Geometry is Hard!



```
%
  Get the screensize.
%
    screen = get ( 0, 'ScreenSize' );
%
%
  Use the whole screen as a figure.
%
  figure ( 'Position', screen );
%
%
  Create a graphical coordinate system on the figure.
%
  axes ( 'Position', [ 0, 0, 1, 1 ] );
%
\% The user clicks sample points on the boundary, termin
%
  [x, y] = ginput ();
```

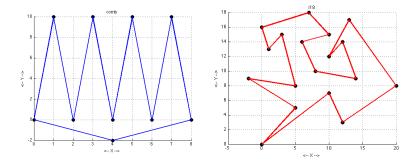
https://people.sc.fsu.edu/~jburkardt/m_src/hand_data/hand_acquire.m

14: From Points We Have an outline



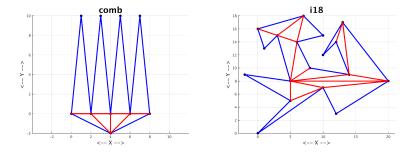
https://people.sc.fsu.edu/~jburkardt/m_src/hand_data/hand_data.png

15: Polygons Are a Little Messy



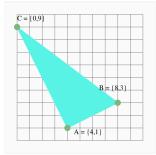
 $\label{eq:https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/comb.png \\ \https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18.png \\ \end{tabular}$

16: But Polygons Can Be Triangulated



 $\label{eq:https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/comb_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulation.png \\ \mbox{https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/i18_triangulate_tes$

17: The Area of a Triangle



area
$$(A, B, C) = 1/2 \begin{vmatrix} B_x - A_x & B_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix}$$

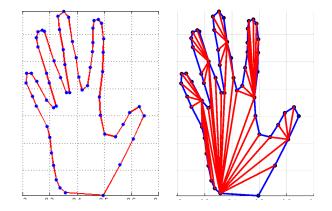
= $1/2((B_x - A_x)(C_y - A_y) - (B_y - A_y)(C_x - A_x))$
= $1/2((8 - 4)(9 - 1) - (3 - 1)(0 - 4))$
= 20

function area = triangle_area (ax, ay, bx, by, cx, cy)

return end

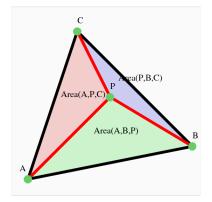
 $https://people.sc.fsu.edu/{\sim}jburkardt/m_src/triangle_properties/triangle_area.m$

19: I Can Compute The Area of My (Flat) Hand



https://people.sc.fsu.edu/~jburkardt/m_src/polygon_triangulate_test/hand_triangulation.png

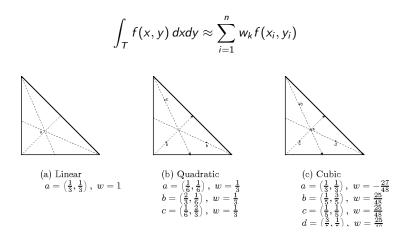
20: Does a Triangle Contain a Point?



P is inside T if all three areas are positive:

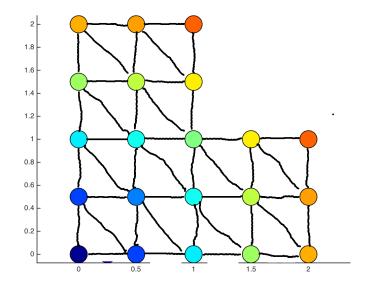
- Area(P,B,C)
- Area(A,P,C)
- Area(A,B,P)

21: Can We Integrate Over a Triangle?

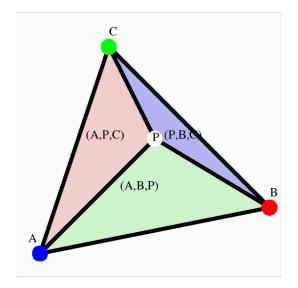


https://people.sc.fsu.edu/~jburkardt/datasets/quadrature_rules_tri/quadrature_rules_tri.html

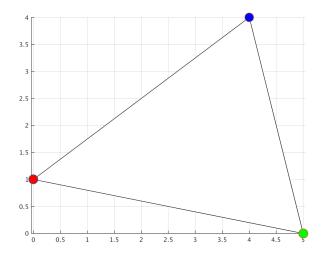
22: Computer Data Only Computed at a Few Points



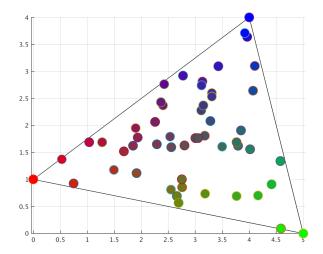
23: Area-Averaging Vertex Values



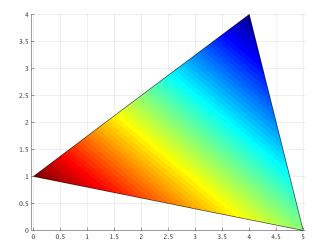
24: We Know the Values at Triangle Vertices



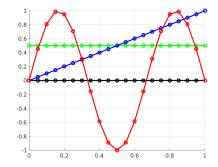
25: Fill Some Points in the Triangle



26: Fill All Points in a Triangle



26.5: Physics is about the bends!



$$\frac{\partial T}{\partial t} = \nabla^2 T$$
$$\nabla T = \frac{\partial T}{\partial x} \approx \frac{T(x+dx) - T(x)}{dx}$$
$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} \approx \frac{T(x+dx) - 2T(x) + T(x-dx)}{dx^2}$$

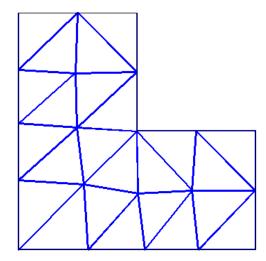
```
[ p, t ] = mesh2d ( v );
```

- v, a list of boundary (x,y) coordinates;
- p, the coordinates of nodes;
- *t*, triples of nodes forming triangles.

https://www.mathworks.com/matlabcentral/fileexchange/25555-mesh2d-automatic-mesh-generation

 $https://people.sc.fsu.edu/{\sim}jburkardt/m_src/mesh2d_test/ell_test.m$

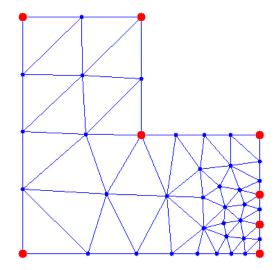
29: MESH2D: Simple ELL Mesh



$$v = \begin{bmatrix} 0.0, 0.0; \dots \\ 2.0, 0.0; \dots \\ 2.0, 0.25; \\ 2.0, 0.5; \dots \\ 2.0, 1.0; \dots \\ 1.0, 1.0; \dots \\ 1.0, 2.0; \dots \\ 0.0, 2.0 \end{bmatrix};$$

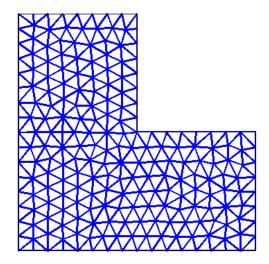
[p, t] = mesh2d (v);

31: MESH2D: Two Short Boundary Segments



[p, t] = mesh2d (v, [], hdata);

33: MESH2D: Set Maximum Element Size



[p, t] = mesh2d (v, [], hdata);

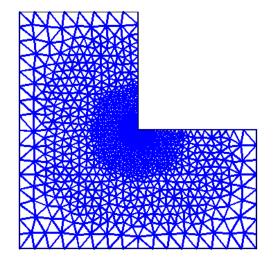
```
function h = hfun (x, y)
```

```
%
%
% Minimum size is 0.01, increasing as we move away
% from ( 1.0, 1.0 ).
%
h = 0.01 + 0.1 * sqrt ( ( x-1.0 ).^2 + ( y-1.0 ).^2 );
```

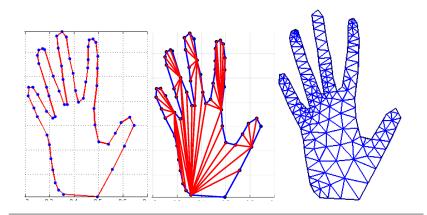
return

end

36: MESH2D: Ask for a Small Mesh Near the Indentation

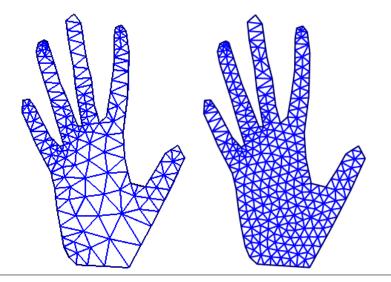


37: Apply MESH2D to the Hand Data



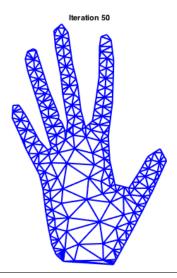
 $https://people.sc.fsu.edu/\sim jburkardt/m_src/hand_mesh2d_test/hand_mesh2d_mesh.png$

38: Apply MESH2D to the Hand Data



 $https://people.sc.fsu.edu/\sim jburkardt/m_src/hand_mesh2d_test/hand_mesh2d_mesh.png$

39: A Mesh That's Finest Near the Edge



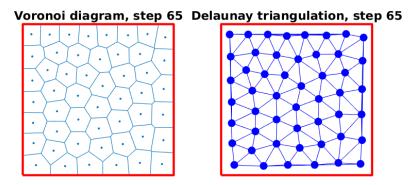
 $https://people.sc.fsu.edu/{\sim}jburkardt/m_src/distmesh_test/p24_mesh.png$

40: A Mesh from Nature



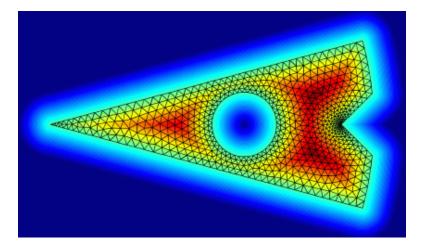
https://en.wikipedia.org/wiki/Centroidal_Voronoi_tessellation

41: CVT: Centroidal Voronoi Tessellation



https://people.sc.fsu.edu/~jburkardt/m_src/cvt_square_uniform_test/cvt_square_uniform.png

42: A CVT Mesh for the Holey Pie

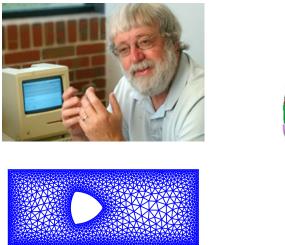


Is there time for a quick movie?

https://people.sc.fsu.edu/~jburkardt/presentations/cvt_movie_p08_cramped.mp4

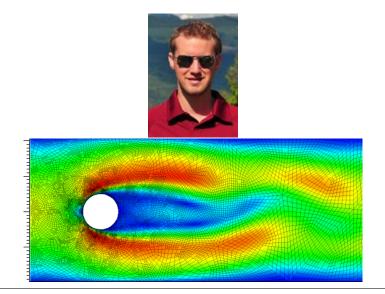
xdg-open cvt_movie_p08_cramped.mp4

44: Meshes that Adapt to Flat or Curved Surfaces



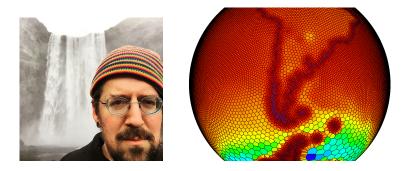
 $https://people.sc.fsu.edu/{\sim}mgunzburger/$

45: 2D: A Moving Mesh for Fluid Flow



https://people.sc.fsu.edu/~lb13f/

47: $2\frac{1}{2}$ D: Accurate Mesh of Coastline Interfaces



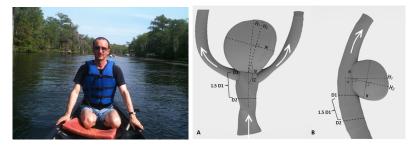
https://www.linkedin.com/in/geoff-womeldorff-030112153/

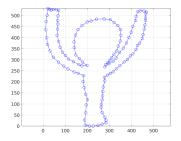
48: $2\frac{1}{2}$ D: Scan, Mesh and Classify Bones for Age-At-Death



https://www.linkedin.com/in/detelina-stoyanova-05912876/

49.5: When Will a Blood Vessel Burst?





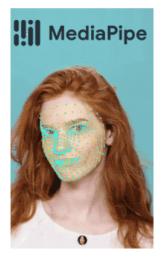


49.6: Computer Games Use Meshes



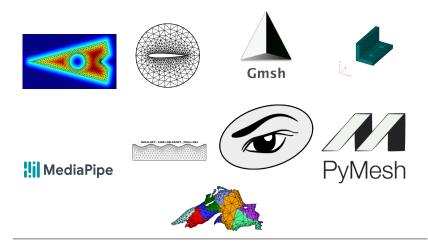
An example mesh editor for the Unity game engine.

49.7: Google MediaPipe



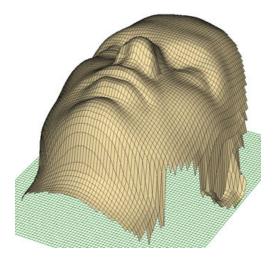
https://google.github.io/mediapipe

49.8: Meshing Programs



distmesh, freefem++, gmsh, matlab, mediapipe, mesh2d, meshlab, pymesh, triangle

50: A Flatbed Scanner Makes a Surface Mesh



https://www.library.pitt.edu/3d-scanning-printing

51: A Wrap-Around Scanner for Head-Protection Studies



https://morphlab.sc.fsu.edu/people/lab_members.html

Is there time for a another movie?

 $https://people.sc.fsu.edu/{\sim}jburkardt/presentations/head_scan_movie.mp4$

xdg-open head_scan_movie.mp4

53: Results of a Wrap-Around Scan



 $https://people.sc.fsu.edu/{\sim}jburkardt/presentations/my_head.ply$

54: Careers for Applied Mathematics Majors

CAREERS in applied mathematics SIAM SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

CAREERS IN APPLIED MATHEMATICS / 5

EDUCATION B.S. Mathematics and Psychology. 2011 Worcester State University M.S. Applied Statistics 2012. Worcester 2012. Worcester

> Early-6 years post Bachelor's

DONT

STRESS OVER

PERFECT JOB

THE GATE

KAITLYN BRADY / DATA SCIENTIST

HIAT SHE DOEL // No bedges are ever the same for Nahn A si data scientiti. It is the responsible to entract and communicate accinetion bright hom data. Here not entering on understanding here company's automem. Who are they? What are their preference of Prozph Inswelder of their customers who are they preference their experiments and in turn increases company performance. Beceful you worked on building appendix the problem, building the algorithms, and writing the code.

NECESSARY 308 SETLES (1 To be a dbb softward you must be well rounded with abboard of the softward softward softward to a softward to use of the softward softward softward abb may be solved on the job to use and that C casels to possible softward is of that Each problem as definers from the lost and wort 11 a testboar mold. Addita scientify must identify, and sometimes involve indications abboard and balance abboard.

Mathematics is important in allowing Kathyn to provide concrete data-supported evidence for the ecommendators and answers the provider to business. Rather than making a subjective juggment: the use stitutics to formulas a solution that is based on reason and bigs. Examples of concepts she has implemented so fair include time areas analysis, logistic regression, and non parametric massures for hypothesis toting.

PROS AND COMS OF HER 308 // One of the more rewarding aspects of Kablyn's job is when she innew that she delivered comething new and empowered a business partner to aslance their campaign or project-i-leveraging computer programming to bring a solution to production in an efficient manner. Rubyn's least Avente thing is day-to day reporting.

CABLED RAYE of Kallyn bound her first full-time job as addas solutions engineer through here is choicil care first in Silver was responsible for a lob care also and was descript and real mailed that she washed to do more administrational work. That is when the movied to Stagies as a statistical stagistic where its because segments the care company analight and any particular stage. After come stagistic where its because segments the care company analight and any particular stage. After come startice tame where the work is more research oriented with borger term graphics faithing care applic contrasts to obser a side learn more stade horiset field with the company.

WITH EACH NEW ROLE I FICK UP ADDITIONAL SKILLS AND PASSIONS AND PUSH NYSELF TOWARDS A MORE DEFINED CAREER.

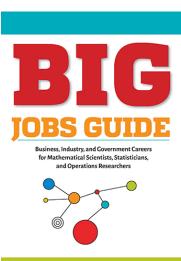
SALARY // An approximate salary range is \$90,000 to \$200,000+ depending on education experience, and location.

CAREER EXPECTATIONS AND ADVICE // internehips and applied projects are great ways for students to gain new world ways internet has well leach them new skills, help them lean what they do and do not like, and the guide unsume. For these just starting their career, don't tries over thinking the particity biright out of the gate—you are still developing and learning and with them pency of opportunities to more around.



https://www.siam.org

55: BIG: Business, Industry, Government



Rachel Levy • Richard Laugesen • Fadil Santosa

https://www.siam.org

SLIDE 1/2

My talk today involves a subject that spreads across several fields of study. Although I enjoy the mathematics that is involved, what I am doing is probably most comfortable as an example of scientific computing. Scientific computing, or computational science, has only recently been recognized as its own area, and now degree programs and separate departments are appearing in colleges across the country.

I'd like to say that if you enjoy mathematics, or computer science or any science, but especially enjoy doing these things when there is a computer involved, and an interesting real life problem to solve, then a program in computational science might be worth considering.

Although my first career choice (age 6) was "farmer", my first sensible choice was "something about mathematics", because I loved reading the Mathematical Games articles by Martin Gardner.

Starting in mathematics, my interest in writing, education, programming and research has led me to Scientific Computing, and it's probably too late now to make any more big changes. Mathematics intrigued me because it showed that simple things like numbers and shapes could actually have mysterious properties.

In high school, I read a math book that described a proof that it was possible to cut a ball up into a few pieces that could be reassembled to make two balls of the same volume as the original one. (This is the Banach-Tarski paradox.)

And I read that equations like y = f(x) could not only describe lines and parabolas and cubics - there was also a formula y = f(x) that corresponded exactly to the profile of my own face.

I decided this was too nonsensical to believe, but I really wanted to hang out with people who thought things up like this!

https://www.quantamagazine.org/how-a-mathematical-paradox-allows-infinite-cloning-20210826/

After getting a mathematics degree, I was working at Iowa State University (half time research, half time running the computer lab) when I walked into a graduate student's office and saw a plot on his computer screen that just didn't make sense to me.

It was beautiful and it was mathematical. That doesn't happen often!

My friend, Lili Ju, now a professor at the University of South Carolina, explained to me that this was actually an example of a kind of mesh that organized itself, almost the way it would happen in nature.

I've got to know more about this! I told him, and I spent some time learning what he had done and how it could be used.

And I think that leads me into my story...

SLIDE 8 / 9 / 10 / 11

The Egyptians used 3-4-5 triangles of rope to resurvey the Nile banks after floods.

The Greeks named geo-metry and trigono-metry, and Eratostenes used this to measure the size of the Earth, the tilt of the Earth's axis, the distance to the sun.

Artists discovered that a complex object could be copied by viewing it through a wire mesh, and then filling in the boxes.

The French revolution invented the meter, and then had to measure a section of longitude in order to figure out what the meter was. ("The Measure of the Earth" by Larrie Ferreiro.)

Mathematicians didn't see any reason to divide smooth space up into triangles, (until much much later), but engineers and computational scientists realized the same lesson that artists had discovered: a complex shape may be easier to work with if a regular mesh is used to break it into many similar pieces.

https://www.amazon.com/Measure-Earth-Enlightenment-Expedition-Reshaped/dp/0465063810

It's pretty easy to see how computers can be taught to count, by turning decimal numbers into binary numbers, binary numbers into strings of 0's and 1's, and 0's and 1's into electronic switches.

But how do I explain the shape of Mickey Mouse's head? The area of lake Erie? The branching pattern of a human lung?

Let's take a shape that's not too simple, but not so complex, and "teach" the computer to see it.

Let's lower our expectations, and assume that the shapes we describe will be drawn using a sequence of straight lines. This means we're working with poly-gons (Greek for "many sides").

Using a polygon, we can still ask many interesting geometric questions, including

- perimeters,
- areas,
- centers of mass,
- whether a point is inside or outside a shape,
- the integral of some function over the triangle $\int_T f(x, y) dx dy$;
- the distance between a point and a shape
- can I get through a maze?
- how many objects can fit inside this shape?
- do these two shapes fit together?

Answering these questions is sometimes easy for us - but how do we teach the computer to do this?

If we have a shape outlined by straight lines, it is always possible to regard it as a collection of triangles.

My hand is an interesting shape (at least to me). Using a program like MATLAB, we can record points on my hand's outline so the computer can "see" what I see.

My hand is represented by a polygon. I don't know a lot about polygons, but I do know a lot about triangles. As it turns out, any polygon can be dissected into triangles, by "slicing off one ear at a time." And you can teach the computer to do this.

Thanks to the Greeks, we know lots of things about triangles, such as the area. There's a formula for the area of any triangle with vertices A, B, C. Using this, I can get the area of (the polygon representing) my hand.

It is possible to get a **negative area**. Indeed, if A = (0,0), B = (1,4), C = (3,2), we get area (A,B,C) = -5. Interestingly enough, area (A,C,B) = +5.

The minus sign is telling us something very useful: the triangle (A,B,C) has its vertices listed in clockwise order, but (A,C,B) lists them in counterclockwise order, The sign of the area is a warning about the orientation of the triangle.

As long as we promise to list triangle vertices in counterclockwise order, we will have no problems with the area formula. But it turns out that this bit of knowledge can be used to determine other information.

Suppose I have a shape that I have turned into a polygonal outline, and then into a collection of triangles.

And now suppose I ask whether the point P is inside the shape? P is inside the shape (or at least the polygon) if it is inside a triangle. And it is inside triangle ABC exactly if all three subtriangles formed by P have positive area.

So that means we can solve the "point inside shape" geometry problem.

Many physical laws can be expressed in terms of integrals. Problems in heat conduction, fluid flow, elasticity, can be written this way. They can be solved if we can estimate integrals over the triangles of a mesh.

The finite element method rewrites a physical law in integral form, redraws the physical region as a mesh of triangles, estimates integrals over the triangles, and produces an approximate solution for the problem.

To approximate an integral over a triangle, there is an extensive family of quadrature rules that explain how to sample the function and weight the results.

 $https://people.sc.fsu.edu/\sim jburkardt/datasets/quadrature_rules_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/quadratures_tri/qua$

We measure temperature at a few places, but ask for its value anywhere.

To make a good estimate, we need to take the nearest data and somehow spread it to the query point.

If our data is on the vertices of triangles, then for any point P, we can find the triangle containing P, and use the triangle average of the vertex data. That is, the value at P is constructed by using the values at A, B, and C in the proportions of the triangles PBC, APC, and ABP.

Our estimates are most accurate if the triangles are regular shaped and relatively small.

Triangulating the hand completely covers the internal area, but it does so with triangles of many different sizes and shapes.

There might be reasons that we want a pattern of triangles that covers the region more evenly in shape and size.

mesh2d is a computer program for which the user only has to describe an outline of the region of interest, that is, a counterclockwise list of points.

We will start by asking for a simple mesh of a simple region, and then push the code little by little to harder tasks.

Describing the boundary of our region is easy.

mesh2d creates a simple triangular mesh.

If we add two points to the boundary, $\mbox{mesh2d}$ takes the hint and includes more triangles in that area.

We can even specify a maximum triangle size so that **mesh2d** will fill up the region as we wish.

We can even specify that the triangle size changes in a way that suits us. Here, a related program called **distmesh** makes sure we have very small triangles near the inner corner of the region. We can do similar operations on the hand data.

mesh2d creates a simple triangular mesh, with many internal vertices.

We can specify a maximum triangle size of 0.025 inches.

We can specify that triangles should be small near the edges, using a feature that **mesh2d** provides.

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The "computer flower" I saw on Lili Ju's computer screen is an example of a special mesh called a Centroidal Voronoi Tessellation (CVT), a favorite research topic of FSU professor Max Gunzburger.

In nature, for example, the tilapia form nests on a sandy river bottom, and somehow manage to make a mesh there.

To come up with a corresponding computer algorithm, you can imagine letting loose a swarm of bees into a small room, and assume every bee wants to avoid the other bees, and the walls, as much as possible. The resulting pattern is a CVT.

We often want meshes that are fine in some areas and coarse in others. That feature can be included in the CVT.

CVT's are computed using iteration; that is, we make an initial guess for the mesh, and then repeatedly improve it. Here's a short movie that suggests how this works.

Professor Max Gunzburger investigated the creation and use of these special meshes for regions, surfaces, and volumes.

FSU graduate student Lukas Bystricky studied the flow of fluids past obstacles. Behind the obstacle, the fluid whips up and down, and Lukas makes a mesh that follows the fluid action.

FSU postdoc Mauro Perego (now at Sandia lab) studied the slow flow of the gigantic ice sheet over Greenland. In order to get good results, he had to use detailed information about the coastline, about the height of the land surface below the ice, and about the average ice sheet velocity at each point. FSU graduate student Geoff Womeldorff (now at Los Alamos lab) created meshes on the land and ocean, with very small elements where land and ocean overlap, in order to do climate studies.

Using this idea, FSU graduate student Detelina Stoyanova estimated the age of skeletons by scanning a particular bone, creating a mesh of triangles, which could then be compared against an online collection of such scans.

Undergraduates Marcelina Nagales and Alexa Pennavaria used the scanner to start a scientific scheme for classifying early North American arrowheads by style, region, and age.

Pitt Professor Catalin Trenchea is working on a project which is trying to give some mathematical insight into a deadly medical problem known as an aneurysm. When a blood vessel wall develops a weak spot, the high interior pressure can gradually push out a balloon-shaped distortion. At some point the stretched surface can break suddenly, releasing an uncontrolled flow of blood into the body or the brain.

This study is looking for ways of understanding the stresses on the surface wall, the different kinds of shapes that can develop, and the signs that might tell whether an aneurysm is simply a mild distortion, or a serious danger that may break soon.

As part of the study, sample diagrams of aneurysms must be outlined, and meshed, allowing a model of the internal blood flow and pressure to be developed. FSU Professor Dennis Slice ran a geometric morphology lab. He has always had a 2D scanner, but now he has a 3D head scanner, for research on helmet design. FSU graduate student Alex Townsend sat me down in the barber's chair and promised me it wouldn't hurt.

A laser measures hundreds of thousands of points on the head.

A computer program can then organize these points into a triangular mesh representing the shape of the head.

You may remember that I mentioned at the beginning that mathematics promised me that an equation for the profile of my face "existed", although it offered no way of discovering it. Computational Science has produced a very reasonable result.

It's may be just an approximation, but now I can bet my head on it!

One place to get a realistic view of job opportunities for an applied mathematics major is the web site for SIAM (the Society for Industrial and Applied Mathematics). Their brocure on "Careers in Applied Mathematics" includes profiles of people in many careers, how they got there, what they do, and what they are earning. You need to start thinking about life after school, and this brochure is one way to get some ideas of the choices available to you.

Most job opportunities for mathematics majors are NOT in the university, but rather in business, industry, and government. These fields can be unfamiliar to anyone who has spent their life in school. SIAM publishes a friendly, information book, "The BIG Job Guide", which goes over the kind of jobs available, how to apply for them, what to expect if you get called in for an interview, what your working days will be like if you get hired, and the chances for promotion, salary, travel, and further training.