Sensitivity Inconsistency for Geometric Parameters

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In this study, we examine the relationship between the exact derivatives of a discrete state variable with respect to some parameter, and the discretized sensitivities. The exact derivatives may be defined as the solution of the sensitivity equation derived from the discrete state equation. By contrast, the discretized sensitivities are derived by discretizing the sensitivity equation of the continuous state equation.

For many types of parameters, the operations of differentiation (computing the sensitivity equation) and discretization can be interchanged. However, this need not be the case for parameters which control geometric quantities, such as the location or shape of the boundary, an interface, or an obstacle. In some cases, the discrepancy can be quite considerable.

We consider a model problem, involving the steady flow of a viscous incompressible fluid moving through a rectangular channel with a partial obstruction. One set of parameters controls the strength of the inflow function, while another determines the shape of the obstruction, which lies along the bottom of the channel.

The governing state equations are the time-independent Navier Stokes equations with the continuity equation. The two sets of parameters exert their influence in the form or location of some of the associated boundary conditions.

We assume sufficient smoothness of the dependence of the flow solution on the parameters so that we can differentiate the state variables with respect to the parameters, and, if desired, interchange differentiation with respect to a parameter and with respect to space. Under these assumptions, if we have computed a solution to the continuous state equations, we can derive an appropriate sensitivity equation for that solution. Generally this is done by simple differentiation of the state equations, but the process is more complicated when geometric parameters are involved.

In practice, of course, it is out of the question to compute a solution to an arbitrary Navier Stokes system, and so some form of discretization is required, so that an approximate solution can be computed. We apply the finite element method to our continuous problem. For any set of parameters, the corresponding discrete flow solution can easily be computed. Under the same smoothness assumptions, the discrete flow solution can be regarded as a differentiable function of the parameters, and a sensitivity equation can be derived by differentiating the finite element equations with respect to the parameter of interest.

Direct differentiation of the finite element equations for the inflow parameter is surprisingly simple. The differentiation operator "commutes" with the finite element integral operator, and so we may actually regard this process as equivalent to the finite element method applied to the sensitivity equation for the continuous problem. In other words, the same basic algorithm can be applied to solving both the state equation and the sensitivity equation.

For a geometric parameter, however, this is not necessarily the case. In fact, for the model problem, direct differentiation of the finite element equations results in many new terms that were not present in the discretization of the state equations. This is because the region of integration is affected by changes in the parameter. Moreover, the location of finite element grid points and the actual form of the finite element basis functions can also depend on such a parameter, leading to a very cumbersome form for the differentiated discrete state equation.

Thus, for such a parameter, it is tempting to prefer the computation of discretized sensitivities by applying the finite element method to the sensitivity equation of the continuous problem. This attack has the advantage that, as in the case of the inflow parameter, the same algorithm can be applied to solve both the state and sensitivity equations. A typical discretized sensitivity field for a parameter controlling the shape of a bump is displayed in Figure **??**. The sensitivities may be regarded as the flow solution of the homogeneous Oseen equations. The boundary conditions along the bump then completely determine the flow.

A number of problems arise, though, in this approach. First, it should be clear that there is no guarantee that the computed discretized sensitivities are the derivatives of the discrete state quantities. Instead, the two sets of data are related only indirectly: the continuous state variables are approximated by the discrete state variables; the sensitivities of the continuous state variables are approximated by the discretized sensitivities.

One may invoke appropriate approximation theorems to assert that, as the mesh size decreases, the discretized sensitivities will, in fact, become arbitrarily good approximations of the sensitivities of the discrete state quantities. However, for a given problem and fixed mesh size, the errors committed may be unacceptably large.

The source of this error can, in part, be traced to the necessity of defin-

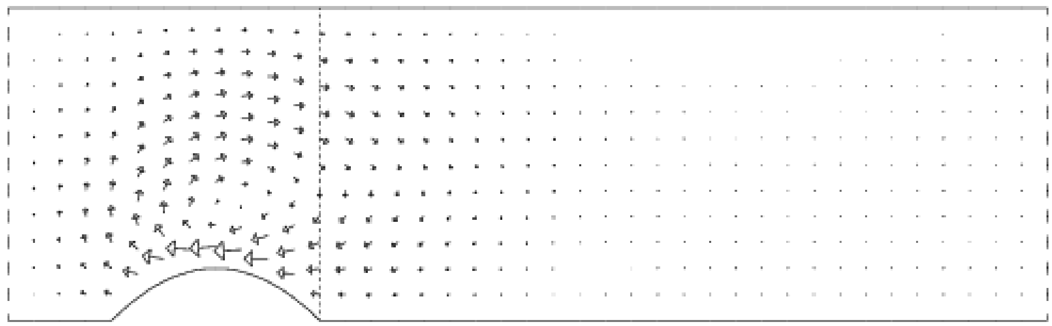

Figure 1: Discretized velocity sensitivity field with respect to the bump. The vertical line to the right of the bump is used to sample the velocities.

ing boundary conditions for the discretized sensitivities. To do so correctly, the exact values of partial spatial derivatives of the continuous state variables along portions of the boundary are needed. We lose considerable accuracy when we actually form the boundary conditions since we must use data that has a diminished approximating power. That is, we work at the boundary, where the approximating power of the finite element method is weaker, and at that boundary we estimate derivatives of the state solution, which are more poorly approximated than the actual values of the state solution.

For the model problem, the poor estimation of the appropriate boundary condition is the sole reason for the discrepancy between the correct sensitivities and the discretized sensitivities. We investigate methods of estimating this inaccuracy. We also consider ways of overcoming the problem by attempting to produce higher order accuracy of approximation along the appropriate portion of the boundary.

The original need for sensitivities came about from the need to optimize a cost functional associated with the fluid flow. This functional measured the discrepancy between the computed flow and a desired flow along some vertical line. To properly optimize this functional, it is necessary to compute or estimate the gradient of the cost functional with respect to the various free parameters.

Discretized sensitivities were used to estimate the cost gradient. In several cases where the geometric parameter was allowed to vary, the discretized sensitivities had such significant inaccuracies that the approximate gradient field was inconsistent with the cost functional. A step in the computed direction of descent would actually produce an increase in the cost functional. A typical situation is shown in Figure **??**, where the use of discretized sensitivities to compute the cost gradient has led to the breakdown of the optimization.

We therefore conclude our study by comparing the behavior of our opti-

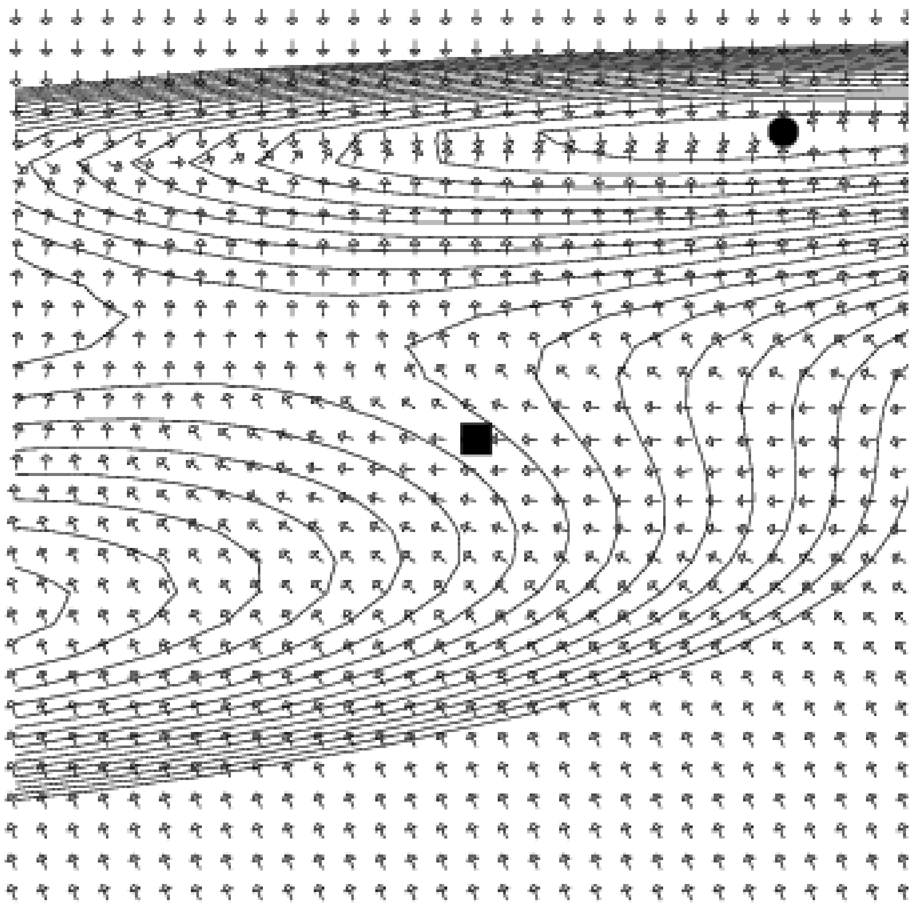

Figure 2: Contours and the approximated gradient field of $\mathcal{J}_2^h$.
Gradients are approximated by discretized sensitivities.
The global minimizer is at the black dot.
The computed minimizer is at the black square.

mization efforts when the cost gradient is computed using correct sensitivities or discretized sensitivities. We then show the improved behavior of the optimization when the computation of the discretized sensitivities is carried out with greater accuracy.