Mitchell Problem #5

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1 Problem

$$-p\frac{\partial^2 u}{\partial x^2} - q\frac{\partial^2 u}{\partial y^2} = f$$

with boundary condtions:

$$p\frac{\partial u}{\partial x}\frac{\partial y}{\partial s} - q\frac{\partial u}{\partial y}\frac{\partial x}{\partial s} + cu = g$$

where p, q, c, f and g are piecewise constant functions over our domain.

2 Weak Formulation

Multiplying by test function ϕ and integrating over Ω yields:

$$-\int_\Omega \left(p\frac{\partial^2 u}{\partial x^2} + q\frac{\partial^2 u}{\partial y^2}\right)\phi d\Omega = \int_\Omega f\phi d\Omega$$

We can integrate by parts:

$$\int_\Omega p\frac{\partial^2 u}{\partial x^2}\phi = p\left(\int_\Gamma u_x\phi d\Gamma - \int_\Omega u_x\phi_x d\Omega\right)$$
$$\int_\Omega q\frac{\partial^2 u}{\partial y^2}\phi = q\left(\int_\Gamma u_y\phi d\Gamma - \int_\Omega u_y\phi_y d\Omega\right)$$

Therefore the weak form can expressed as:

$$\int_\Omega pu_x\phi_x + qu_y\phi_y d\Omega - \int_\Gamma (pu_x + qu_y)\,\phi d\Gamma = \int_\Omega f\phi d\Omega$$

3 Boundary Conditions

Consider a rectangular domain. We can write the boundary integral for each of the 4 sides:

1. **bottom**: here $\frac{\partial x}{\partial s} = 1$ and $\frac{\partial y}{\partial s} = 0$. Then the boundary condtion simplifies to:

$$-q\frac{\partial u}{\partial y} + cu = g \Rightarrow u_y = \frac{cu}{q} - \frac{g}{q}$$

   and the boundary integral can be written as:

$$\int_\Gamma \left(pu_x + q\left(\frac{cu}{q} - \frac{g}{q}\right)\right)\phi d\Gamma = \int_\Gamma (pu_x + cu - g)\,\phi d\Gamma$$

2. **right**: $\frac{\partial x}{\partial s} = 0$ and $\frac{\partial y}{\partial s} = 1$

$$\Rightarrow u_x = \frac{g}{p} - \frac{cu}{p}$$
$$\int_\Gamma (pu_x + qu_y)\,\phi d\Gamma = \int_\Gamma (g - cu + qu_y)\,\phi d\Gamma$$

3. **top**: $\frac{\partial x}{\partial s} = -1$ and $\frac{\partial y}{\partial s} = 0$

$$
\begin{aligned}
\Rightarrow u_y &= \frac{g}{q} - \frac{cu}{q} \\
\int_\Gamma (pu_x + qu_y)\,\phi d\Gamma &= \int_\Gamma (pu_x + g - cu)\,\phi d\Gamma
\end{aligned}
$$

4. **left**: $\frac{\partial x}{\partial s} = 0$ and $\frac{\partial y}{\partial s} = -1$

$$
\begin{aligned}
\Rightarrow u_x &= \frac{cu}{p} - \frac{g}{p} \\
\int_\Gamma (pu_x + qu_y)\,\phi d\Gamma &= \int_\Gamma (cu - g + qu_y)\,\phi d\Gamma
\end{aligned}
$$