Numerical Methods for the Stochastic Nonlocal Model – Peridynamics Model for Mechanics

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Abstract

Mathematical modeling and computer simulations are nowadays widely used tools for the prediction of the behavior of scientific and engineering systems. Our interest is in how uncertainty propagates through the recently developed nonlocal, derivative free, continuum peridynamics model for material mechanics. In contrast to classical partial differential equation models, peridynamics is an integro-differential equation that does not involve spatial derivatives of the displacement field. We focus on the peridynamics model whose forcing terms are described by a finite-dimensional random vector, which is often called finite-dimensional noise assumption. Numerical methods based on this stochastic peridynamics model are implemented. Preliminary results for the one-dimensional problem will be provided and compared.

The Stochastic Peridynamics Model

- **The general bond-based model**

  The equation of motion at any point $\mathbf{x}$ at time t is given by:

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_{\mathbf{x}}} \mathbf{f}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t),\mathbf{x}'-\mathbf{x})\mathrm{d}V_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \quad (1)$$

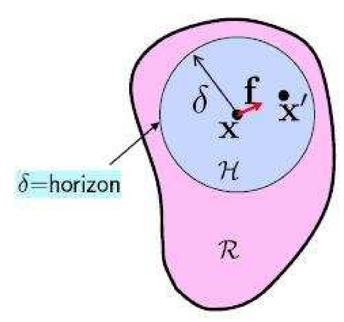
  where ρ – the mass density function,

  $\mathbf{u}$ – the displacement vector field,

  $H_{\mathbf{x}}$ – the neighborhood of $\mathbf{x}$ with radius δ,

  $\mathbf{b}$ – the prescribed body force density field,

  $\mathbf{f}$ – the pairwise function represents the interaction between particles.



- **A linearized peridynamics model for proportional microelastic materials** is given by the integro-differential equation

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_{\mathbf{x}}} c\frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t))\mathrm{d}V_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \quad (2)$$

  where c denotes a constant that depends not only on the material, but also on the space dimension. In one dimension, $c = \frac{18k}{5\delta^2}$, where k denotes the bulk modulus.

- **The stochastic peridynamics model for proportional microelastic materials** in time-independent case is posed as: find a random function, $u : D\times\Omega \to \mathbb{R}$, such that P-almost everywhere (a.e.) in Ω, or in other words, almost surely (a.s.) the following equation holds:

$$\mathcal{L}\mathbf{u}(\mathbf{x},\omega) \equiv \int_{H_{\mathbf{x}}} c\frac{(\mathbf{x}-\mathbf{x}')\otimes(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}(\mathbf{u}(\mathbf{x},\omega)-\mathbf{u}(\mathbf{x}',\omega))\mathrm{d}V_{\mathbf{x}'} = \mathbf{b}(\mathbf{x},\omega) \quad (3)$$

Finite-Dimensional Noise Representation

In most applications, the source of randomness for the forcing term $\mathbf{b}(\mathbf{x},\omega)$ appearing in (3) can be described by a random vector $\mathbf{Y} = [Y_1,...,Y_M] : \Omega \to \mathbb{R}^M$, that is

$$\mathbf{b}_M(\mathbf{x},\omega) = \mathbf{b}(\mathbf{x},Y_1(\omega),...,Y_M(\omega)) \quad \text{on} \quad D\times\Omega.$$

where $\{Y_n\}_{n=1}^M$ are real valued random variables with zero mean value and unit variance. Denote by $\Gamma_n \equiv Y_n(\Omega)$ the image, bounded or unbounded, of Y_n and assume that the components of the random vector $\mathbf{Y} = [Y_1,Y_2,...,Y_M]$ have a joint probability density function (PDF) $\rho : \Gamma_M \to R^+$ with $\rho \in L^\infty(\Gamma_M)$, where $\Gamma_M = \prod_{n=1}^M \Gamma_n$, contain the support of this probability density, then the stochastic boundary problem (3) has a deterministic equivalent as follow: seek a random field $\mathbf{u}_M : D\times\Gamma_M \to \mathbb{R}$, such that a.s.,

$$\mathcal{L}\mathbf{u}_M(\mathbf{x},\mathbf{y}) = \mathbf{b}_M(\mathbf{x},\mathbf{y}), \quad \forall\mathbf{x}\in D, \mathbf{y}\in\Gamma_M. \quad (4)$$

Then the corresponding variational formulation is: find $\mathbf{u}_M : \Gamma_M \to \mathbf{W}(D)$ such that

$$\left(\mathcal{L}\mathbf{u}_M(\mathbf{y}),\mathbf{v}\right)_{\mathbf{L}^2(D)} = \left(\mathbf{b}_M(\mathbf{y}),\mathbf{v}\right)_{\mathbf{L}^2(D)}, \quad \forall\mathbf{v}\in\mathbf{W}(D),\ \rho - a.e. \text{ in } \Gamma_M \quad (5)$$

Numerical Methods

- **Monte Carlo Finite Element Method (MCFEM):**
  - **Step 1:** Choose a number of realization, $K \in \mathbb{N}_+$, and a finite element space on D, $\mathbf{W}^h(D)$. For each $j = 1,...,K$, sample iid realizations of the load $\mathbf{b}_M(\omega_j,\cdot)$ and find an approximation $\mathbf{u}_M^h(\omega_j;\cdot) \in \mathbf{W}^h(D)$ such that

$$\left(\mathcal{L}\mathbf{u}_M^h(\omega_j,\cdot),\mathbf{v}\right)_{L^2(D)} = \left(\mathbf{b}_M(\omega_j,\cdot),\mathbf{v}\right)_{L^2(D)} \quad \forall\mathbf{v}\in\mathbf{W}^h(D) \quad (6)$$

  - **Step 2:** Approximate $\mathbb{E}[\mathbf{u}_M](\cdot)$ by the sample average:

$$\mathbb{E}(\mathbf{u}_M^h;K) \equiv \frac{1}{K}\sum_{j=1}^K \mathbf{u}_M^h(\omega_j;\cdot). \quad (7)$$

- **Stochastic collocation method (SC):**
  - **Full tensor product polynomial approximation**
  - **Sparse tensor product polynomial approximation (Smolyak Method)**

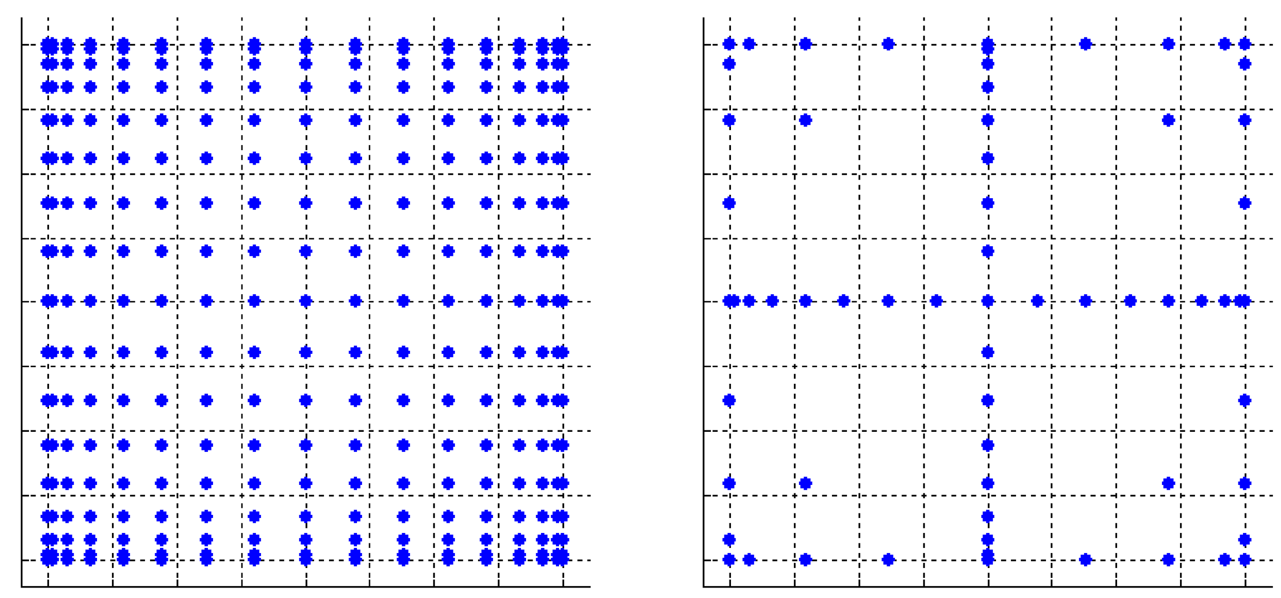


Figure 2: For a finite dimensional Γ_M with $M = 2$ and maximum level $\omega = 4$ we plot the full tensor product grid ($17\times 17 = 289$ points) using the Clenshaw-Curtis abscissas (left) and isotropic Smolyak sparse grids (65 points) $\mathscr{H}(4,2)$, utilizing the Clenshaw-Curtis abscissas (right).

Preliminary Results

Some preliminary results are listed below for the one-dimensional stochastic peridynamics problem (8), i.e.

$$\begin{cases} \dfrac{1}{\delta^2}\displaystyle\int_{x-\delta}^{x+\delta}\frac{u(x,w)-u(x',w)}{|x-x'|}\mathrm{d}x' = b(x,\omega), & x\in(0,1) \\ u(x,w) = g(x,\omega), & x\in[-\delta,0]\cup[1,1+\delta]. \end{cases} \quad (8)$$

where, $b(x,\omega) = b(x) + \sum_{n=1}^M C_n(x)\exp(Y_n(\omega)), Y_n(w)$ are iid $\sim N(0,1)$, $b(x) = 1, g(x,\omega) = x(1-x), C_n(x) = 1/100, \delta = 3h, h = 1/8$.

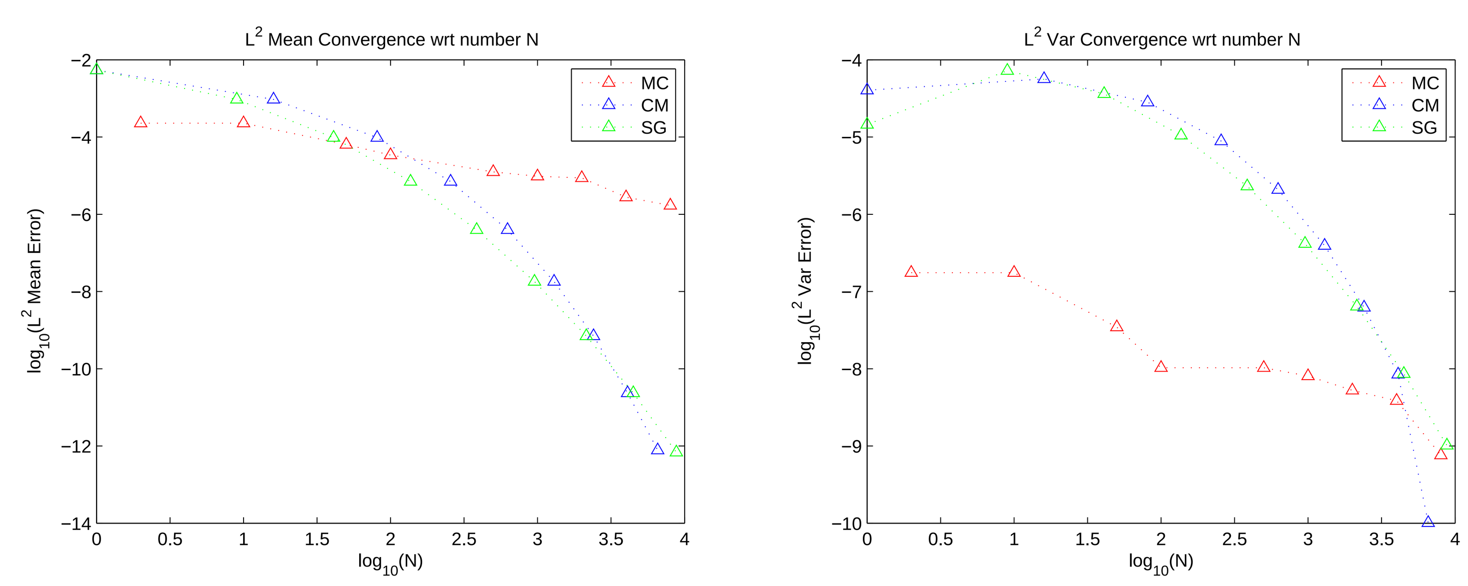


Figure 3: L_2 relative error of mean vs. the number of points for realization for $M = 4$ (left) and L_2 relative error of variance vs. the number of points for realization for $M = 4$ (right).

Future Works

- Implementation for deterministic/stochastic problems in 2 dimensions.
- Complete, rigorous analyses of errors and convergence rates and of adaptive grid refinement strategies for deterministic problems.

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